SHAPE PROPERTIES OF THE SPACE OF PROBABILITY MEASURES AND ITS SUBSPACES

In this article we consider covariant functors acting in the category of compacts, preserving the shapes of infinite compacts, ANR-systems, moving compacts, shape equivalence, homotopy equivalence and \(A(N)SR\) properties of compacts. As well as shape properties of a compact space \(X\) consisting of connectedness components 0 of this compact \(X\) under the action of covariant functors, are considered. And we study the shapes equality \(ShX = ShY\) of infinite compacts for the space \(P(X)\) of probability measures and its subspaces.

**Key words:** Covariant functors, \(A(N)R\)-compacts, ANR-systems, probability measures, moving compacts, retracts, measures of finite support, shape equivalence, homotopy equivalence.


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For a compact \(X\) by \(P(X)\) denote the space of probability measures. It is known that for an infinite compact \(X\), this space \(P(X)\) is homeomorphic to the Hilbert cube \(Q\). For a natural number \(n \in N\) by \(P_n(X)\) denote the set of all probability measures with no more than \(n\) support, i.e. \(P_n(X) = \{\mu \in P(X) : \text{supp}\mu \leq n\}\). The compact \(P_n(X)\) is a convex linear combination of Dirac measures in the form

\[
\mu = \sum_{i=1}^{n} m_i \delta_{x_i}, \quad \sum_{i=1}^{n} m_i = 1, m_i \geq 0, x_i \in X,
\]

\(\delta_{x_i}\) is the Dirac measure at a point \(x_i\). By \(\delta(X)\) denote the set of all Dirac measures. Recall that the space \(P_f(X) \subset P(X)\) of all probability measures in the form \(\mu = \sum_{i=1}^{n} m_i \delta_{x_i} + m_2 \delta_{x_2} + \ldots + m_n \delta_{x_n}\) of finite support, for each of which \(m_i \geq \frac{k}{i+1}\) for some \(i\). For a positive integer \(n\) put \(P_{f,n} = P_f \cap P_n\). For a compact \(X\) we have \(P_{f,n} = \{\mu \in P_f(X) : \text{supp}\mu \leq n\}\), \(P_{f} \equiv P_{f,0} \cap P_{C}\), \(P_{f,n} \equiv P_{f} \cap P_{n} \cap P_{C}\), \(P_{f,n} \equiv P_{f} \cap P_{n} \cap P_{C}\). For the compact \(X\) by \(P_{C}\) denote the set of all measures \(\mu \in P(X)\) the support of each of which lies in one of the components of the compact \(X\) [12].

1. Introduction

For a space \(X\) by \(\square X\) denote the expansion (partition) of the space \(X\) consisting of all the connected components. If \(f : X \to Y\) is a continuous mapping, then the continuous mapping \(\square f : \square X \to \square Y\) is uniquely determined by condition \(\pi_Y \circ f = \square f \circ \pi_X\), where \(\pi_Y : Y \to \square Y\) and \(\pi_X : X \to \square X\), i.e. we have the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\pi_X \downarrow & \downarrow \square f & \downarrow \pi_Y \\
\square X & \to & \square Y \\
\end{array}
\]  

(1.1)

**Lemma 1.** If \(X\) is a compact ANR-space, then the map \(P_{C}(\pi_X)\) isahomotopy equivalence.
Proof. Let be an ANR-compact, then the space $P^C(X)$ is a finite set or is a finite union of Hilbert cubes and points. The space $P^C(\square X)$ consists of finitely many points, because the space $X$ is an ANR-compact. For any $\mu \in P^C(\square X)$ the transformation $(P^C(f))^{-1}(\mu)$ is the Hilbert cube, or one point, i.e. $Sh((P^C(f))^{-1}(\mu))$ is trivial, then by Theorem 7 [5] the map $P^C(f)$ is a shape equivalence, and thus, is a homotopy equivalence. The proof is complete.

Theorem 1. Let $X$ be a compact and let $\pi_X : X \to \square X$ be a quotient map. Then the mapping $P^C(\pi_X)$ induces a shape equivalence, i.e. $Sh(P^C(X)) = Sh(\square X)$.

Proof. Suppose $X$ is compact, $\square X$ is also compact, then by V.I. Ponomarev theorem [6] $\dim \square X = 0$. Hence, $\dim P^C(X) = 0$ and $P^C(\square X) = 0$. By Theorem 2 [5] the mapping $P^C(\pi_X)$ is a shape equivalence. This means that $Sh(P^C(X)) = ShP^C(\square X)$ and $\lVert P^C(\square X) \rVert = \lVert \square X \rVert$. This proves the theorem.

Definition [10]. A normal subfunctor $F$ of the functor $P_n$ is called locally convex if the set $F(\tilde{n})$ is locally convex.

We say that a functor $F_1$ is a subfunctor (respectively nadfunktorom?) of a functor $F_2$ if there exists a natural transformation $h : F_1 \to F_2$ that the map $h(X) : F_1(X) \to F_2(X)$ is a monomorphism (epimorphism) for each object $X$. From the identity functor $Id$ is a subfunctor of $exp_n$, where $exp_n X = \{ F \in \exp X : |F| \leq n \}$, and the $n$th degree functor $n$ is a nadfunktorom of functors $exp_n$ and $SP^C_n$. A normal subfunctor $F$ of the functor $P_n$ is uniquely determined by its value $F(n)$ at an $n$-point space. Note that $P_n(n)$ is the $(n-1)$-dimensional simplex. Any subset of the $(n-1)$-dimensional simplex $\sigma^{n-1}$ defines a normal subfunctor of the functor $P_n$ if it is invariant under simplicial mappings.

An example of not normal subfunctor of the functor $P_n$ is the functor of probability measures $P^n_C$ whose supports lie in one of components. One of the examples of locally convex subfunctors of $P_n$ is a functor $SP^n \equiv SP^n _{Sh}$.

Corollary 1. If for compacts $X$ and $Y$ the equality $\lVert \square X \rVert = \lVert \square Y \rVert = n_0$ holds, then $Sh(P^C(X)) = ShP^C(\square X)$ and $ShP^C(Y) = ShP^C(\square Y)$, where $|Z|$ is the cardinality of a set $Z$.

Proof. Suppose the sets $\square X$ and $\square Y$ are countable. In this case, by Arkhangelskii’s result [8], the spaces $\square X$ and $\square Y$ are compact and metrizable. Note that $\square X$ and $\square Y$ have a dense set of isolated points. Then the compacts $P(X)$ and $P(Y)$ are homeomorphic to the Hilbert cube $Q$. On the other hand, $P^C(\square X) = \square X$ and $P^C(\square Y) = \square Y$. Consequently, $Sh(P^C(\square X)) = Sh(P^C(\square Y))$. The corollary is proved.

By $M_\square$ we denote the class of all compacts $X$ such that $\square X$ is metrizable. From corollary it follows that if $X, Y \in M_\square$, then $\square X$ and $\square Y$ have a countable dense set of isolated points [9].

Corollary 2. If $X, Y \in M_\square$, then either $Sh(P^C(X)) \geq Sh(P^C(Y))$ or $Sh(P^C(X)) \leq Sh(P^C(Y))$.

Therefore, if $\square X$ and $\square Y$ are infinite, then $Sh(P^C(X)) = Sh(P^C(Y))$, i.e. $Sh(P^C(X)) \geq Sh(P^C(Y))$ and $Sh(P^C(X)) \leq Sh(P^C(Y))$.

Proof. Suppose that $X$ and $Y$ are elements of the family $M_\square$. Then $\square X$ and $\square Y$ are the zero-dimensional compacts. In particular, if $\square X$ and $\square Y$ are finite sets, then by Theorem 1 we obtain the desired.

If $\lVert \square X \rVert \geq n_0$, then $\square X$ contains Cantor’s discontinuum. In this case, $\square Y$ can be embedded into $\square X$, then the compact $\square Y$ is a retract for $\square X$ [10]. $Sh(\square X) \geq Sh(\square Y)$ and $Sh(P^C(\square X)) \geq Sh(P^C(\square Y))$.

Consequently, by Theorem 1 we have $ShP^C(\square X) \geq ShP^C(\square Y)$, if $\square X \leq n_0$ and $\square Y \leq n_0$, then compacts $\square X$ and $\square Y$ are homeomorphic to Mazurkiewicz-Sierpinski ordinal compact [11]. Last, suppose $\square X$ and $\square Y$ are infinite sets, then $Sh(\square X) \geq Sh(\square Y)$ if and only if $\square X$ and $\square Y$ are homeomorphic [3]. If $\lVert \square X \rVert > \lVert \square Y \rVert$ or $\lVert \square X \rVert < \lVert \square Y \rVert$, then either $\square Y$ or $\square X$ is retract for $\square X$ or $\square Y$, respectively. By Theorem 1 we have $Sh(P^C(X)) \geq Sh(P^C(Y))$. Corollary 2 is proved.

Remark. In [11] it is shown that the Borsuk’s definition of shapes of compacts is equivalent to the shapes of ANR-systems.

Lemma 2. For any compact $X$ we have $\lVert \square P_f(X) \rVert = \lVert \square X \rVert$.

Proof. Let $X$ be an arbitrary compact, $\square X$ its set of connected components, i.e. $\square X = \{ \pi^X(x) \}$ is connected component of the point $x$. It is obvious that $\square X$ is compact and $\square X \subset X$. Hence, $Sh(\square X) \leq ShX$. On the other hand, the commutativeness of the diagram

$$
\begin{align*}
\pi_X : X &\to \square X \\
&\uparrow \\
P_f(\pi_X) : P_f(X) &\to \delta(\square X)
\end{align*}
$$

implies $\lVert \square ShP_f(X) \rVert = \lVert \square X \rVert$. From (1.2) we get $\lVert \square P_f(X) \rVert = \lVert \square X \rVert$. Lemma 2 is proved.
Let us note that for all \( x \in X \) and \( y \in X \) between sets \( \left(r^{-1}_f(x)\right) \) and \( \left(r^{-1}_f(y)\right) \) there is a one-one correspondence, i.e. to an arbitrary point \( \mu_x \in \left( P^{-1}_f \right)(X) \) we assign \( \mu_y \in \left( P^{-1}_f \right)(Y) \), where

\[
\mu_x = m_0 \delta_{x_0} + m_1 \delta_{x_1} + ... + m_k \delta_{x_k}, \mu_y = m_0 \delta_{y_0} + ... + m_k \delta_{y_k}.
\]

In the case of the infinite sets \( X \) and \( Y \) the spaces \( P(X) \) and \( P(Y) \) are homeomorphic to the Hilbert cube \( Q \). If \( A \) and \( B \) are \( Z \)-sets lying in the compacts \( P(X) \) and \( P(Y) \), then by Chapman’s theorem [2], \( ShA = ShB \) if and only if \( P(X) \setminus A \) is homeomorphic to \( P(Y) \setminus B \). In [10,12] it is shown that the subspaces \( F(X) \) and \( F(Y) \) are \( Z \)-sets in the compacts \( P(X) \) and \( P(Y) \), where \( F = P_{f,n}(X), P_{f,n}(Y) \), \( P_{f,n}^C(X), P_{f,n}^C(Y) \). Moreover, it was noted that this space \( X \) is a strong deformation retract for \( F(X) \). So the following is valid.

**Theorem 2.** For infinite compacts \( X \) and \( Y \) the following conditions are equivalent:

1. \( ShX = ShY \);
2. \( P(X) \setminus P_f(X) \simeq P(Y) \setminus P_f(Y) \);
3. \( P(X) \setminus \delta(X) \simeq P(Y) \setminus \delta(Y) \);
4. \( P(X) \setminus F(X) \simeq P(Y) \setminus F(Y) \), where \( F = P_{f,n}^C, P_{f,n}^C \).

**Theorem 3.** Suppose that \( X \) and \( Y \) are elements of \( M_\square \), \( X \in M_\square \) and \( Y \in M_\square \). Then the following conditions are equivalent:

1. \( Sh(\square X) = Sh(\square Y) \);
2. \( P(X) \setminus P^C(X) \simeq P(Y) \setminus P^C(Y) \).

**Theorem 4.** Suppose that \( X \) and \( Y \) are elements of \( M_\square \). Then \( Sh(\square X) = Sh(\square Y) \) if and only if \( ShX = Sh(\square X) \).

It is known that from the inequality \( ShX \leq ShY \) it follows \( Sh(\square X) \leq Sh(\square Y) \). In particular, the equality \( ShX = ShY \) implies \( Sh(\square X) = Sh(\square Y) \).

Now let \( Sh(\square X) = Sh(\square Y) \). From the fact that the compacts \( \square X \) and \( \square Y \) are zero-dimensional and metrizable, and by Mardeschicha Segal theorem [3], \( \square X \) and \( \square Y \) are homeomorphic. If for any \( y \in \square X \) the set \( \pi^{-1}_y(y) \) has the trivial shape, then by Theorem 7 [5] we have \( ShY = Sh(\square Y) \); by virtue of the zero-dimensionality and equality \( ShX = Sh(\square X) \) it follows \( Y \simeq \square X \simeq \square Y \).

Note that in this case \( ShX = ShY \) and \( X \simeq Y \), i.e. \( ShX = Sh(\square X) \) is equivalent to \( ShX = ShY \).

**Corollary 3.** a) The space \( P^C(X) \) is an \( ASR \) if and only if \( X \) is connected; b) \( P^C(X) \) is an \( ANSR \) if and only if \( X \) has finitely many connected components.

**Theorem 5.** For any infinite zero-dimensional compacts \( X \) and \( Y \) the followings are true:

a) If \( ShX = Sh Y \), then \( P_n(X) \simeq P_n(Y) \);

b) if \( ShX \neq Sh Y \), then \( P(X) \setminus P_n(X) \simeq P(Y) \setminus P_n(Y) \);

c) \( ShP_n(X) = ShP_n(Y) \) if and only if \( P(X) \setminus P_n(X) \simeq P(Y) \setminus P_n(Y) \);

d) \( ShF(X) = ShF(Y) \) if and only if \( P(X) \setminus F(X) \simeq P(Y) \setminus F(Y) \), where \( F \) are locally convex subfunctors of the functor \( P_n \);

e) \( ShX = ShY \) if and only if \( P(X) \setminus \delta(X) \simeq P(Y) \setminus \delta(Y) \).

**Theorem 6.** For any infinite zero-dimensional compacts \( X \) and \( Y \) the following conditions are equivalent:

1. \( ShX = ShY \);
2. \( ShF(X) = ShF(Y) \), where \( F = P_{f,n}, P_{f,n}^C, P_f, P_f^C \);
3. \( X \simeq Y \);
4. \( P(X) \setminus F(X) \simeq P(Y) \setminus F(Y) \).

**Theorem 7.** For any infinite compacts \( X \) and \( Y \) we have: a) if \( ShX = Sh Y \), then \( P(X) \setminus P_n(X) \simeq P(Y) \setminus P_n(Y) \) for any \( n \in N \);

b) if \( ShX = Sh Y \), then \( P(X) \setminus F(X) \simeq P(Y) \setminus F(Y) \), where \( F \) are locally convex subfunctors of the functors \( P_n \).

**Theorem 8.** For any infinite compacts \( X \in M_\square \) and \( Y \in M_\square \) we have:

a) \( ShX = ShY \) if and only if \( P(X) \setminus P_n(X) \simeq P(Y) \setminus P_n(Y) \);

b) \( ShX = ShY \) if and only if \( P(X) \setminus F(X) \simeq P(Y) \setminus F(Y) \).
References


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СВОЙСТВА ФОРМЫ ВЕРОЯТНОСТНОГО ПРОСТРАНСТВА И ЕГО ПОДПРОСТРАНСТВ

В этой заметке мы рассматриваем ковариантные функторы, действующие в категории компактов, сохраняющие формы бесконечных компактов, ANR-системы, движущиеся компакты, эквивалентность формы, гомотопическую эквивалентность и A(N)SR свойства компактов. Рассмотрены свойства формы компактного пространства X, состоящего из компонент связности 0 этого компактного X под действием ковариантных функторов. И мы изучаем равенство форм ShX = ShY бесконечных компактов для пространства вероятностных мер P(X) и его подпространства.

Ключевые слова: Ковариантный функтор, шейп компакта, компонента, связности и гомотопическая эквивалентность.


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