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SYMMETRIC FINITE REPRESENTABILITY OF ℓ^p IN ORLICZ SPACES[1]

ABSTRACT
It is well known that a Banach space need not contain any subspace isomorphic to a space ℓ^p $(1 \leqslant p < \infty)$ or c^0 (it was shown by Tsirel'son in 1974). At the same time, by the famous Krivine's theorem, every Banach space X always contains at least one of these spaces *locally*, i.e., there exist finite-dimensional subspaces of X of arbitrarily large dimension n which are isomorphic (uniformly) to ℓ_p^n for some $1 \leqslant p < \infty$ or c_0^n. In this case one says that ℓ^p (resp. c^0) is finitely representable in X. The main purpose of this paper is to give a characterization (with a complete proof) of the set of p such that ℓ^p is *symmetrically* finitely representable in a separable Orlicz space.

Key words: ℓ^p-space; finite representability of ℓ^p-spaces; symmetric finite representability of ℓ^p-spaces; Orlicz function space; Orlicz sequence space; Matuszewska-Orlicz indices.

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Introduction

While a Banach space X need not contain any subspace isomorphic to a space ℓ^p $(1 \leqslant p < \infty)$ or c^0 (as was shown by Tsirel'son in [1]), it will always contain at least one of these spaces *locally*. This means that there exist finite-dimensional subsets of X of arbitrarily large dimension n which are isomorphic (uniformly) to ℓ_p^n for some $1 \leqslant p < \infty$ or c_0^n. This fact is the content of the famous result proved by Krivine in [2] (see also [3]). To state it we need some definitions.

Suppose X is a Banach space, $1 \leqslant p \leqslant \infty$, and $\{z_i\}_{i=1}^\infty$ is a bounded sequence in X. The space ℓ^p is said to be *block finitely representable in* $\{z_i\}_{i=1}^\infty$ if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $0 = m_0 < m_1 < \ldots < m_n$ and $\alpha_i \in \mathbb{R}$ such that the vectors $u_k = \sum_{i=m_{k-1}+1}^{m_k} \alpha_i z_i$, $k = 1, 2, \ldots, n$, satisfy the inequality

$$(1+\varepsilon)^{-1}\|a\|_p \leqslant \left\| \sum_{k=1}^n a_k u_k \right\|_X \leqslant (1+\varepsilon)\|a\|_p$$

for arbitrary $a = (a_k)_{k=1}^n \in \mathbb{R}^n$. In what follows,

$$\|a\|_p := \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \text{ if } p < \infty, \text{ and } \|a\|_\infty := \max_{k=1,2,\ldots,n} |a_k|$$

The space ℓ^p, $1 \leqslant p \leqslant \infty$, is said to be *finitely representable* in X if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $x_1, x_2, \ldots, x_n \in X$ such that for any $a = (a_k)_{k=1}^n \in \mathbb{R}^n$

$$(1+\varepsilon)^{-1}\|a\|_p \leqslant \left\| \sum_{k=1}^n a_k x_k \right\|_X \leqslant (1+\varepsilon)\|a\|_p$$

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(alternatively, in the case $p = \infty$, one might say that c^0 is finitely representable in X).

Clearly, if ℓ^p is block finitely representable in some sequence $\{z_i\}_{i=1}^\infty \subset X$, then ℓ^p is finitely representable in X. Therefore, the following famous result proved by Krivine in [2] (see also [3] and [4, Theorem 11.3.9]) implies the finite representability of ℓ^p for some $1 \leqslant p \leqslant \infty$ in any Banach space.

Theorem (Krivine)

Let $\{z_i\}_{i=1}^\infty$ be an arbitrary normalized sequence in a Banach space X such that the vectors z_i do not form a relatively compact set. Then ℓ^p is block finitely representable in $\{z_i\}_{i=1}^\infty$ for some $p \in [1, \infty]$.

Here, we consider both Orlicz sequence and function spaces (see the next section for the definition) and in the separable case we give a characterization of the set of p such that ℓ^p is *symmetrically finitely representable* in such a space. To introduce the notion of symmetric finite representability, we need some more definitions.

A sequence $y = (y_k)_{k=1}^\infty$ will be called a *copy* of a sequence $x = (x_k)_{k=1}^\infty$ if x and y have the same entries, that is, there is a permutation π of the set of positive integers such that $y_{\pi(k)} = x_k$ for all $k = 1, 2, \ldots$.

Given a measurable function $x(t)$ on $[0, 1]$, we set

$$n_x(\tau) := m(\{t \in [0, \alpha) : |x(t)| > \tau\}), \quad \tau > 0.$$

Here and in the sequel, m denotes the Lebesgue measure. Functions $x(t)$ and $y(t)$ are called *equimeasurable* if $n_x(\tau) = n_y(\tau)$ for each $\tau > 0$.

Let X be a symmetric sequence space (see e.g. [5]), $1 \leqslant p \leqslant \infty$. We say that ℓ^p is *symmetrically finitely representable* in X if for every $n \in \mathbb{N}$ and each $\varepsilon > 0$ there exists an element $x_0 \in X$ such that for its disjoint copies x_k, $k = 1, 2, \ldots, n$, and for every $(a_k)_{k=1}^n \in \mathbb{R}^n$ we have

$$(1+\varepsilon)^{-1}\|a\|_p \leqslant \Big\| \sum_{k=1}^n a_k x_k \Big\|_X \leqslant (1+\varepsilon)\|a\|_p$$

Similar notion will be defined also in the function case. Let X be a symmetric function space on $[0, 1]$ [5]. The space ℓ^p is *symmetrically finitely representable* in X if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist equimeasurable and disjointly supported on $[0, 1]$ functions $u_i(t)$, $i = 1, 2, \ldots, n$, such that for all $(a_k)_{k=1}^n \in \mathbb{R}^n$

$$(1-\varepsilon)\|a\|_p \leqslant \Big\| \sum_{i=1}^n a_i u_i \Big\|_X \leqslant (1+\varepsilon)\|a\|_p$$

The set of all $p \in [1, \infty]$ such that ℓ^p is symmetrically finitely representable in X (in both sequence and function cases) we will denote by $\mathcal{F}(X)$.

From the definition[2] of the Matuszewska-Orlicz indices α_N^0 and β_N^0 (resp. α_N^∞ and β_N^∞) of an Orlicz sequence space ℓ_N (resp. an Orlicz function space L_N) it follows that $\mathcal{F}(X) \subset [\alpha_N^0, \beta_N^0]$ (resp. $\mathcal{F}(X) \subset [\alpha_N^\infty, \beta_N^\infty]$). The main purpose of this paper is to give a detailed proof of the opposite embedding for both Orlicz sequence and function spaces. To this end, following the idea mentioned in [6, p. 140–141] we will make use of the proof of Theorem 4.a.9 from [7].

Similar problems for Orlicz function spaces (and more generally symmetric spaces) on $(0, \infty)$ were considered in the paper [8].

1. Preliminaries

1.1. Orlicz sequence spaces

A detailed information related to Orlicz sequence and function spaces see in monographs [9–11].

The Orlicz sequence spaces are a natural generalization of the ℓ^p-spaces, $1 \leqslant p \leqslant \infty$, which equipped with the usual norms

$$\|a\|_{\ell^p} := \begin{cases} \left(\sum_{k=1}^\infty |a_k|^p\right)^{1/p}, & 1 \leqslant p < \infty \\ \sup\limits_{k=1,2,\ldots} |a_k|, & p = \infty. \end{cases}.$$

Let N be an Orlicz function, that is, an increasing convex continuous function on $[0, \infty)$ such that $N(0) = 0$ and $\lim_{t\to\infty} N(t) = \infty$. The *Orlicz sequence space* ℓ_N consists of all sequences $a = (a_k)_{k=1}^\infty$, for which the following (Luxemburg) norm

$$\|a\|_{\ell_N} := \inf\left\{ u > 0 : \sum_{k=1}^\infty N\Big(\frac{|a_k|}{u}\Big) \leqslant 1 \right\}$$

[2] See the next section.

is finite. Without loss of generality, we will assume that $N(1)=1$. In particular, if $N(t)=t^p$, we get the ℓ^p-space, $1\leqslant p<\infty$.

Recall that the Matuszewska-Orlicz indices (at zero) α_N^0 and β_N^0 of an Orlicz function N are defined by

$$\alpha_N^0 := \sup\big\{p : \sup_{x,y\leqslant 1}\frac{N(x)y^p}{N(xy)}<\infty\big\},\qquad \beta_N^0 := \inf\big\{p : \inf_{x,y\leqslant 1}\frac{N(x)y^p}{N(xy)}>0\big\}.$$

It can be easily checked that $1\leqslant\alpha_N^0\leqslant\beta_N^0\leqslant\infty$. It is well known also that an Orlicz sequence space ℓ_N is separable if and only if $\beta_N^0<\infty$, or equivalently, if the function N satisfies the Δ_2-condition at zero, i.e.,

$$\limsup_{u\to 0}\frac{N(2u)}{N(u)}<\infty.$$

The subset h_N of an Orlicz sequence space ℓ_N consists of all $(a_k)_{k=1}^\infty\in\ell_N$ such that

$$\sum_{k=1}^\infty N\Big(\frac{|a_k|}{u}\Big)<\infty\ \text{ for each } u>0.$$

One can easily check (see also [7, Proposition 4.a.2]) that h_N is a separable closed subspace of ℓ_N and the canonical unit vectors $e_n=(e_n^i)$ such that $e_n^n=1$ and $e_n^i=0$ if $i\neq n$, $n=1,2,\ldots$, form a symmetric basis of the space h_N. Recall that a basis $\{x_n\}_{n=1}^\infty$ of a Banach space X is said to be *symmetric* if there exists $C>0$ such that for any permutation π of the set of positive integers and all $a_n\in\mathbb{R}$ we have

$$C^{-1}\Big\|\sum_{n=1}^\infty a_nx_n\Big\|_X\leqslant\Big\|\sum_{n=1}^\infty a_nx_{\pi(n)}\Big\|_X\leqslant C\Big\|\sum_{n=1}^\infty a_nx_n\Big\|_X.$$

Observe that the definition of an Orlicz sequence space ℓ_N is determined (up to equivalence of norms) by the behaviour of the function N near zero. More precisely, the following conditions are equivalent: 1) $\ell_N=\ell_M$ (with equivalence of norms); 2) the canonical vector bases of the spaces h_N и h_M are equivalent; 3) there are constants $C>0$, $c>0$ and $t_0>0$ such that for all $0\leqslant t\leqslant t_0$ it holds

$$cN(C^{-1}t)\leqslant M(t)\leqslant c^{-1}N(Ct)$$

(see e.g. [7, Proposition 4.a.5] or [11, Theorem 3.4]). In particular, if N is a degenerate Orlicz function, i. e., for some $t_0>0$ we have $N(t)=0$ if $0\leqslant t\leqslant t_0$, then $\ell_N=l_\infty$ (with equivalence of norms).

Given Orlicz function N, we define the following subsets of the space $C[0,\frac12]$:

$$E_{N,a}^0 := \overline{\Big\{\frac{N(xy)}{N(y)}\ : 0<y<a\Big\}},\quad E_N^0 := \bigcap_{0<a<1}E_{N,a}^0$$

and

$$C_{N,a}^0 := \overline{conv E_{N,a}^0},\quad C_N^0 := \bigcap_{0<a<1}C_{N,a}^0,$$

where $0<a<1$ and the closure is taken in the norm topology of $C[0,\frac12]$. All these sets are non-void norm compact subsets of the space $C[0,\frac12]$ [7, Lemma 4.a.6]. It is well known that they determine to a large extent the structure of disjoint sequences of Orlicz sequence spaces (see [7, § 4.a] and [12]). Moreover, if $1\leqslant p<\infty$, then $t^p\in C_N^0$ if and only if $p\in[\alpha_N^0,\beta_N^0]$ [7, Theorem 4.a.9].

In the case when an Orlicz function N satisfies the Δ_2-condition at zero, the sets $E_{N,a}^0$, E_N^0, $C_{N,a}^0$ and C_N^0 can be considered as subsets of the space $C[0,1]$ (see the remark after Lemma 4.a.6 in [7]).

1.2. Orlicz function spaces

Let N be an Orlicz function such that $N(1)=1$. Denote by L_N the Orlicz space on $[0,1]$ endowed with the Luxemburg norm

$$\|x\|_{L_N} := \inf\{u>0 : \int_0^1 N\Big(\frac{|x(t)|}{u}\Big)\,dt\leqslant 1\}.$$

In particular, if $N(t)=t^p$, $1\leqslant p<\infty$, we obtain the space $L_p=L_p[0,1]$ with the usual norm.

The Matuszewska-Orlicz indices α_N^∞ and β_N^∞ (at infinity) of an Orlicz function N are defined by the formulae

$$\alpha_N^\infty = \sup\big\{p : \sup_{x,y\geqslant 1}\frac{N(x)y^p}{N(xy)}<\infty\big\},\qquad \beta_N^\infty = \inf\big\{p : \inf_{x,y\geqslant 1}\frac{N(x)y^p}{N(xy)}>0\big\}.$$

Again $1 \leqslant \alpha_N^\infty \leqslant \beta_N^\infty \leqslant \infty$. As in the case of sequence spaces, an Orlicz space L_N is separable if and only if $\beta_N^\infty < \infty$, or equivalently, if the function N satisfies the Δ_2-*condition at infinity*, i.e.,

$$\limsup_{u\to\infty} \frac{N(2u)}{N(u)} < \infty.$$

In contrast to the sequence case, the definition of an Orlicz function space L_N on $[0,1]$ is determined (up to equivalence of norms) by the behaviour of the function $N(t)$ for large values of t.

For every Orlicz function N we define the following subsets of the space $C[0,\frac{1}{2}]$:

$$E_{N,A}^\infty := \overline{\{\frac{N(xy)}{N(y)} \;: y > A\}}, E_N^\infty = \bigcap_{A>0} E_{N,A}^\infty, \quad C_N^\infty := \overline{convE_N^\infty},$$

where the closure is taken in the norm topology of $C[0,\frac{1}{2}]$. Again all these sets are non-void norm compact subsets of the space $C[0,\frac{1}{2}]$ and they determine largely the structure of disjoint sequences in Orlicz function spaces (see [12, Propositions 3 and 4]). Moreover, if $1 \leqslant p < \infty$, then $t^p \in C_N^\infty$ if and only if $p \in [\alpha_N^\infty, \beta_N^\infty]$ [12].

Finally, if an Orlicz function N satisfies the Δ_2-condition at infinity, the sets $E_{N,A}^\infty$, E_N^∞ and C_N^∞ can be considered as subsets of the space $C[0,1]$.

2. Symmetric finite representability of ℓ^p in Orlicz sequence spaces

Theorem 1

Let M be an Orlicz function satisfying the Δ_2-condition at zero. Then ℓ^p is symmetrically finitely representable in the Orlicz sequence space ℓ_M if and only if $p \in [\alpha_M^0, \beta_M^0]$, i.e., $\mathcal{F}(\ell_M) = [\alpha_M^0, \beta_M^0]$.

Proof.

As was observed in § 1, we always have $\mathcal{F}(\ell_M) \subset [\alpha_M^0, \beta_M^0]$. Therefore, it suffices to prove only the opposite embedding. In other words, we need to show that for every $p \in [\alpha_M^0, \beta_M^0]$, $m \in \mathbb{N}$ and each $\varepsilon > 0$ there exists an element $x_0 \in \ell_M$ such that for its disjoint copies x_k, $k = 1, 2, \ldots, m$, and for every $c = (c_k)_{k=1}^m \in \mathbb{R}^n$ we have

$$(1+\varepsilon)^{-1}\|c\|_p \leqslant \Big\| \sum_{k=1}^m c_k x_k \Big\|_{\ell_M} \leqslant (1+\varepsilon)\|c\|_p. \tag{1}$$

According to the proof of Theorem 4.a.9 in [7] and a comment followed this proof on p. 144, $t^p \in C_M^0$ (see also § 2.1). Since M satisfies the Δ_2-condition at zero, the set C_M^0 may be considered as a subset of the space $C[0,1]$ (see the remark after Lemma 4.a.6 in [7] or again § 2.1). Therefore, since $C_M^0 := \bigcap_{0<a<1} C_{M,a}^0$, we conclude that $t^p \in C_{M,2^{-n}}^0$ for each $n \in \mathbb{N}$.

Note that the mapping

$$\lambda \mapsto M_\lambda(t) := M(\lambda t)/M(\lambda) \tag{2}$$

is continuous from $I_n := (0, 2^{-n}]$ into the subset $E_{M,2^{-n}}^0$ of $C[0,1]$. Indeed, as it is well known (see e.g. [9, Theorem 1.1]),

$$M(t) = \int_0^t \rho(s)\,ds, \tag{3}$$

where ρ is a nondecreasing right-continuous function.

Therefore, for arbitrary $\lambda_2 > \lambda_1 > 0$ and all $0 \leqslant t \leqslant 1$ we have

$$\begin{aligned} |M_{\lambda_2}(t) - M_{\lambda_1}(t)| &= \frac{|M(\lambda_1)M(\lambda_2 t) - M(\lambda_2)M(\lambda_1 t)|}{M(\lambda_1)M(\lambda_2)} \\ &\leqslant \frac{1}{M(\lambda_2)}(M(\lambda_2 t) - M(\lambda_1 t) + M(\lambda_2) - M(\lambda_1)) \\ &\leqslant \frac{1}{M(\lambda_2)}\Big(\int_{\lambda_1 t}^{\lambda_2 t} \rho(s)\,ds + \int_{\lambda_1}^{\lambda_2} \rho(s)\,ds\Big) \\ &\leqslant \frac{2\rho(\lambda_2)}{M(\lambda_2)}(\lambda_2 - \lambda_1). \end{aligned}$$

Thus, mapping (2) may be extended uniquely to a map $\omega \mapsto M_\omega$ from the Stone-Čech compactification βI_n of I_n onto the set $E_{M,2^{-n-1}}^0$. Since $t^p \in C_{M,2^{-n}}^0$ and the extreme points of $C_{M,2^{-n}}^0$ are contained in the compact set $E^0_{M,2^{-n}}$, by the Krein-Milman theorem (see e.g. [13, Theorem 3.28]), there exists a probability measure μ_n on the set βI_n such that

$$t^p = \int_{\beta I_n} M_\omega(t)\, d\mu_n(\omega), \quad 0 \leqslant t \leqslant 1. \tag{4}$$

Let us show that
for some probability measure ν_n on I_n we have

$$\left| t^p - \int_0^{2^{-n}} M_\lambda(t)\, d\nu_n(\lambda) \right| < 2^{-n}, \quad 0 \leqslant t \leqslant 1. \tag{5}$$

First, the fact that the set $\mathbb{Q}_n := \mathbb{Q} \cap I_n$ ($\mathbb{Q}$ is the set of rationals) is dense in βI_n implies that the set $\{M_r, r \in \mathbb{Q}_n\}$ is dense in the subset $\{M_\omega, \omega \in \beta I_n\}$ of $C[0,1]$. Consequently, putting $\mathbb{Q}_n = \{r_k\}_{k=1}^\infty$ and

$$E_k := \{\omega \in \beta I_n : |M_\omega(t) - M_{r_k}(t)| < 2^{-n} \text{ for all } 0 \leqslant t \leqslant 1\}, \tag{6}$$

we have $\beta I_n = \bigcup_{k=1}^\infty E_k$. Now, if $F_m := E_m \setminus (\bigcup_{k=1}^{m-1} E_k)$, $m = 1, 2, \ldots$, then F_m are pairwise disjoint and $\beta I_n = \cup_{m=1}^\infty F_m$. Define the measure ν_n on σ-algebra of Borel subsets U of the interval I_n by

$$\nu_n(U) := \sum_{\{k:\, r_k \in U\}} \mu_n(F_k),$$

where μ_n is the probability measure from (4). Since

$$\nu_n(I_n) = \sum_{k=1}^\infty \mu_n(F_k) = \mu_n(\beta I_n) = 1,$$

then ν_n is a probability measure on I_n. Moreover, by (4) and (6), for all $0 \leqslant t \leqslant 1$

$$\begin{aligned}
\left| t^p - \int_0^{2^{-n}} M_\lambda(t)\, d\nu_n(\lambda) \right| &= \left| \int_{\beta I_n} M_\omega(t)\, d\mu_n(\omega) - \int_0^{2^{-n}} M_\lambda(t)\, d\nu_n(\lambda) \right| \\
&\leqslant \sum_{k=1}^\infty \left| \int_{F_k} M_\omega(t)\, d\mu_n(\omega) - \int_{\{r_k\}} M_\lambda(t)\, d\nu_n(\lambda) \right| \\
&\leqslant \sum_{k=1}^\infty \left| \int_{\{r_k\}} (M_\lambda(t) + 2^{-n})\, d\nu_n(\lambda) - \int_{\{r_k\}} M_\lambda(t)\, d\nu_n(\lambda) \right| \\
&\leqslant 2^{-n} \sum_{k=1}^\infty \nu_n(\{r_k\}) = 2^{-n} \nu_n(I_n) = 2^{-n-1},
\end{aligned}$$

and inequality (5) is proved.

Next, for any $s \in (0,1)$ and $n, j \in \mathbb{N}$ we set

$$a_{j,n} := \int_{s^j 2^{-n}}^{s^{j-1} 2^{-n}} \frac{d\nu_n(\lambda)}{M(\lambda)}. \tag{7}$$

Then, by inequality (5), we have

$$\sum_{j=1}^\infty [a_{j,n}] M(s^j 2^{-n} t) - 2^{-n} < t^p < \sum_{j=1}^\infty [a_{j,n}] M(s^{j-1} 2^{-n} t) + M(t) 2^{-n}/(1-s) + 2^{-n},$$

where by $[z]$ we denote the integer part of a real number z. Choosing now k_n such that

$$\sum_{j=k_n+1}^\infty [a_{j,n}] M(s^{j-1} 2^{-n}) < 2^{-n},$$

as $M(t) \leqslant M(1) = 1$, we get

$$F_n(st) - 2^{-n+1} < t^p < F_n(t) + 2^{-n}/(1-s) + 2^{-n+1}, \quad 0 \leqslant t \leqslant 1, \tag{8}$$

where

$$F_n(t) := \sum_{j=1}^{k_n} [a_{j,n}] M(s^{j-1} 2^{-n} t). \tag{9}$$

Since the right derivative ρ of M (see (3)) is a nondecreasing function and $0 < s < 1$, from (7) it follows that

$$\begin{aligned} F_n(t) - F_n(st) &\leqslant \sum_{j=1}^{k_n} a_{j,n}(M(s^{j-1}2^{-n}t) - M(s^j 2^{-n}t)) \\ &\leqslant \sum_{j=1}^{k_n} \frac{2^{-n}s^{j-1}(1-s)\rho(s^{j-1}2^{-n})}{M(s^j 2^{-n})} \int_{s^j 2^{-n}}^{s^{j-1}2^{-n}} d\nu_n(\lambda). \end{aligned}$$

Furthermore, the estimate

$$F(2x) \geqslant \int_x^{2x} \rho(s)\, ds \geqslant x\rho(x), \ \ 0 \leqslant x \leqslant 1,$$

combined with the hypothesis that M satisfies the Δ_2-condition at zero, shows that

$$K := \sup_{0<x\leqslant 1} \frac{x\rho(x)}{M(x)} < \infty.$$

Hence,

$$F_n(t) - F_n(st) \leqslant K(1-s) \sum_{j=1}^{k_n} \frac{M(s^{j-1}2^{-n})}{M(s^j 2^{-n})} \int_{s^j 2^{-n}}^{s^{j-1}2^{-n}} d\nu_n(\lambda).$$

Moreover, one can readily check that the upper Matuszewska-Orlicz index β_M^0 is finite (see also § 2.1) and, by its definition, for each $q > \beta_M$ there is a constant $c_0 > 0$ such that

$$M(s^j 2^{-n}) \geqslant c_0 M(s^{j-1}2^{-n}) s^q.$$

As a result, since ν_n is a probability measure, we conclude

$$F_n(t) - F_n(st) \leqslant K(1-s)s^{-q}c_0^{-1} \sum_{j=1}^{k_n} \int_{s^j 2^{-n}}^{s^{j-1}2^{-n}} d\nu_n(\lambda) \leqslant K(1-s)s^{-q}c_0^{-1}. \tag{10}$$

Let $m \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Choose and fix $s \in (0,1)$ so that

$$K(1-s)s^{-q}c_0^{-1} < \varepsilon/(2m). \tag{11}$$

Then, from (8) and (10) it follows

$$F_n(t) - \frac{\varepsilon}{2m} - 2^{-n+1} < F_n(st) - 2^{-n+1} < t^p, \ \ 0 \leqslant t \leqslant 1. \tag{12}$$

Now, taking $n \in \mathbb{N}$ satisfying the inequality

$$\frac{2^{-n}}{1-s} + 2^{-n+1} < \frac{\varepsilon}{2m}, \tag{13}$$

from (8) and (12), we obtain

$$F_n(t) - \frac{\varepsilon}{m} < t^p < F_n(t) + \frac{\varepsilon}{m}, \ \ 0 \leqslant t \leqslant 1. \tag{14}$$

Therefore, for any $c_i \in [0,1]$, $i = 1, 2, \ldots, m$,

$$\sum_{i=1}^m c_i^p - \varepsilon < \sum_{i=1}^m F_n(c_i) < \sum_{i=1}^m c_i^p + \varepsilon,$$

whence for all $c = (c_k)_{k=1}^n \in \mathbb{R}^n$, $c_k \geqslant 0$,

$$1 - \varepsilon < \sum_{i=1}^m F_n\left(\frac{c_i}{\|c\|_p}\right) < 1 + \varepsilon.$$

Moreover, since F_n is a convex function, from the latter inequality it follows that

$$\sum_{i=1}^m F_n\left(\frac{c_i}{(1+\varepsilon)\|c\|_p}\right) \leqslant 1$$

and

$$\sum_{i=1}^m F_n\left(\frac{c_i}{(1-\varepsilon)\|c\|_p}\right) > 1.$$

Therefore, by the definition of the norm in an Orlicz sequence space, for every $m \in \mathbb{N}$ and all $c = (c_k)_{k=1}^n \in \mathbb{R}^n$ we have

$$(1-\varepsilon)\|c\|_p \leqslant \Big\|\sum_{i=1}^m c_i e_i\Big\|_{\ell_{F_n}} \leqslant (1+\varepsilon)\|c\|_p, \tag{15}$$

where e_i, $i = 1, 2, \ldots$, are the canonical unit vectors in ℓ_{F_n}.

Given $m \in \mathbb{N}$ and $\varepsilon > 0$, select s and n to satisfy (11) and (13). For any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_n$ denote by $A^i_{j,n}$ pairwise disjoint subsets of positive integers such that $\operatorname{card} A^i_{j,n} = [a_{j,n}]$. Then, the vectors

$$u_i := 2^{-n}\sum_{j=1}^{k_n} s^{j-1}\sum_{k\in A^i_{j,n}} e_k, \ \ i = 1, 2, \ldots, m,$$

are copies of an element from l_m. Moreover, by formula (9), we have

$$\Big\|\sum_{i=1}^m c_i u_i\Big\|_{\ell_M} = \Big\|\sum_{i=1}^m c_i e_i\Big\|_{\ell_{F_n}}$$

for all $c_i \in \mathbb{R}$. Combining this with (15), we get (1), which completes the proof.

3. Symmetric finite representability of ℓ^p in Orlicz function spaces

Theorem 2

Let M be an Orlicz function satisfying Δ_2-condition at infinity. Then ℓ^p is symmetrically finitely representable in the Orlicz function space L_M if and only if $p \in [\alpha_M^\infty, \beta_M^\infty]$, i.e., $\mathcal{F}(L_M) = [\alpha_M^\infty, \beta_M^\infty]$.

Proof.

As in the sequence case, we need only to prove the embedding $[\alpha_M^\infty, \beta_M^\infty] \subset \mathcal{F}(L_M)$. More precisely, we have to check that for every $p \in [\alpha_M^\infty, \beta_M^\infty]$, $m \in \mathbb{N}$ and each $\varepsilon > 0$ there exist equimeasurable and disjointly supported functions u_k, $k = 1, 2, \ldots, m$, satisfying for all $c = (c_k)_{k=1}^m \in \mathbb{R}^m$ the inequality:

$$(1+\varepsilon)^{-1}\|c\|_p \leqslant \Big\|\sum_{k=1}^m c_k u_k\Big\|_{L_M} \leqslant (1+\varepsilon)\|c\|_p \tag{16}$$

First, $t^p \in C_M^\infty \subset C[0,1]$ and then the same reasoning as in the proof of Theorem 1 shows that and that for every $n \in \mathbb{N}$ there is a probabilistic measure ν_n on $[2^n, \infty)$ such that for all $t \in [0,1]$

$$\Big|t^p - \int_{2^n}^\infty \frac{M(\lambda t)}{M(\lambda)}\, d\nu_n(\lambda)\Big| < 2^{-n}.$$

For any $s > 1$ and $n, j \in \mathbb{N}$ we define

$$a_{j,n} := \int_{s^{j-1}2^n}^{s^j 2^n} \frac{d\mu_n(\lambda)}{M(\lambda)}.$$

Then, by the preceding inequality,

$$\sum_{j=1}^\infty a_{j,n} M(s^{j-1}2^n t) - 2^{-n} < t^p < \sum_{j=1}^\infty a_{j,n} M(s^j 2^n t) + 2^{-n}.$$

Next, as M satisfies the Δ_2-condition at infinity, we have

$$M(s^j 2^n t) \leqslant (1+2^{-n}) M(s^{j-1}2^n t)$$

for all $j \in \mathbb{N}$ and $t \in [0,1]$ whenever s is sufficiently close to 1. Fixing such a s, we get

$$\sum_{j=1}^\infty a_{j,n} M(s^{j-1}2^n t) - 2^{-n} < t^p < \sum_{j=1}^\infty (1+2^{-n}) a_{j,n} M(s^{j-1}2^n t) + 2^{-n}.$$

Combining this inequality with the estimate

$$2^{-n}\sum_{j=1}^\infty a_{j,n} M(s^{j-1}2^n t) < 2^{-2n} + 2^{-n}t^p < 2^{-n+1}, \ \ 0 \leqslant t \leqslant 1,$$

we deduce

$$\sum_{j=1}^\infty a_{j,n} M(s^{j-1}2^n t) - 2^{-n} < t^p < \sum_{j=1}^\infty a_{j,n} M(s^{j-1}2^n t) + 2^{-n+2}. \tag{17}$$

On the other hand, since $M(u) \geqslant u$ for all $u \geqslant 1$, we have

$$a_{j,n} \leqslant \frac{2}{M(2^n s^{j-1}} \leqslant 2^{-n} s^{-j+1},$$

which implies that

$$\sum_{j=1}^{\infty} a_{j,n} \leqslant 2^{-n} \sum_{j=1}^{\infty} s^{-j+1} = 2^{-n+1} \cdot \frac{s}{s-1}.$$

Let $m \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Fix n so that

$$\frac{2^{-n+1}s}{s-1} < \frac{1}{m} \quad \text{and} \quad 2^{-n+2}m < \varepsilon. \tag{18}$$

The first of the inequalities (18) allows us to take pairwise disjoint sets $E_j^i \subset [0,1]$, $j \in \mathbb{N}$, $i = 1, 2, \ldots, m$, with $m(E_j^i) = a_{j,n}$. Then, the functions

$$u_i := \sum_{j=1}^{\infty} 2^n s^{j-1} \chi_{E_j^i}$$

are equimeasurable and disjointly supported on $[0,1]$. Moreover, for all $c_i \in \mathbb{R}$

$$\int_0^1 M\Big(\Big|\sum_{i=1}^{m} c_i u_i(t)\Big|\Big)\, dt = \sum_{i=1}^{m} \sum_{j=1}^{\infty} M(2^n s^{j-1}|c_i|) a_{j,n}.$$

Therefore, by (17) and the second inequality in (18), we get

$$\sum_{i=1}^{m} |c_i|^p - \varepsilon < \int_0^1 M\Big(\Big|\sum_{i=1}^{m} c_i u_i(t)\Big|\Big)\, dt < \sum_{i=1}^{m} |c_i|^p + \varepsilon.$$

Repeating further the arguments from the end of the proof of Theorem 1, we come to (16) and so complete the proof.

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СИММЕТРИЧНАЯ ФИНИТНАЯ ПРЕДСТАВИМОСТЬ ℓ^p В ПРОСТРАНСТВАХ ОРЛИЧА[3]

АННОТАЦИЯ

Хорошо известно, что банахово пространство может не содержать подпространств, изоморфных хотя бы одному из пространств ℓ^p $(1 \leqslant p < \infty)$ или c^0 (это было показано Цирельсоном в 1974 г.). В то же время по известной теореме Кривина каждое банахово пространство X всегда содержит хотя бы одно из этих пространств *локально*, т. е. существуют конечномерные подпространства в X сколь угодно большой размерности n, изоморфны (равномерно) ℓ_p^n для некоторых $1 \leqslant p < \infty$ или c_0^n. В этом случае говорят, что ℓ^p (соответственно c^0) финитно представимо в X. Основная цель этой статьи — дать характеризацию (с полным доказательством) множества тех p, что ℓ^p *симметрично* финитно представимо в любом сепарабельном пространстве Орлича.

Ключевые слова: ℓ^p-пространство; финитная представимость ℓ^p-пространств; симметричная финитная представимость ℓ^p-пространств; функциональное пространство Орлича; пространство последовательностей Орлича; индексы Матушевской — Орлича.

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Информация о конфликте интересов: автор и рецензенты заявляют об отсутствии конфликта интересов.

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