Вестник Самарского университета. Естественнонаучная серия. 2020. Том 26, № 1. С. 14–22
Vestnik of Samara University. Natural Science Series. 2020, vol. 26, no. 1, pp. 14–22

Scientific article

DOI: 10.18287/2541-7525-2020-26-1-14-22

Submited: 18.12.2019
Revised: 20.01.2020
Accepted: 28.02.2020

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ON THE SOLUTION OF SOME HIGHER-ORDER INTEGRO-DIFFERENTIAL EQUATIONS OF SPECIAL FORM

ABSTRACT
The article is devoted to the solution of boundary value problems for higher-order linear integro-differential equations of Fredholm type with differential and integral operators encompassing powers of an ideal bijective linear differential operator whose inverse is known explicitly. The conditions for existence and uniqueness of solutions are derived and the solutions are delivered in closed form. The approach is based on the view that an integro-differential operator is a perturbed differential operator. The results obtained are of both theoretical and practical importance. The method is elucidated by solving two illustrative problems.

Key words: integro-differential equations, initial value problems, boundary value problems, differential operators, power operators, composite products, exact solutions.

Citation. Providas E., Parasidis I.N. On the solution of some higher-order integro-differential equations of special form. *Vestnik Samarskogo universiteta. Estestvennonauchnaia seriia = Vestnik of Samara University. Natural Science Series*, 2020, vol. 26, no. 1, pp. 14–22. DOI: http://doi.org/10.18287/2541-7525-2020-26-1-14-22. (In Russ.)

Information about the conflict of interests: authors and reviewers declare no conflict of interests.

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Introduction

Integro-differential equations are used to model physical phenomena and processes in engineering, physics, biology and economics [1–4]. Higher-order Integro-differential equations are also play an important role in mathematical modeling [5–7]. In most cases, they cannot be solved analytically and therefore approximate solutions are persuaded [8; 9]. However, exact solutions are always attractive and in some instances essential [10–12]. Based on the theory of extensions of operators [13–15], the authors have obtained closed form solutions of several types of boundary value problems for integro-differential equations with classical and nonlocal boundary conditions [16–21]. The present paper is a sequel to work of [19] and deals with the exact solution of one more class of boundary value problems of special form.

Let X be a Banach space of complex valued functions of x defined on Ω, $\widehat{A} : X \to X$ a bijective linear differential operator incorporating some initial or boundary conditions. First, we consider the integro-differential equation

$$\begin{aligned} Bu(x) &= \widehat{A}u(x) - \sum_{j=1}^{m} \int_{\Omega} K_j(x,t)\widehat{A}u(t)dt = f(x), \quad x \in \Omega, \\ D(B) &= D(\widehat{A}) \subset X, \end{aligned} \tag{1}$$

where $B : X \to X$ is a linear operator, $K_j(x,t) \in X(\Omega \times \Omega)$ are known kernel functions, $f(x) \in X$ is an input function and $u(x) \in D(B)$ is the sought function describing the response of the system modeled by (1).

Next, we contemplate the more involved integro-differential equation

$$\begin{aligned} B_4u(x) &= \widehat{A}^4u(x) - \sum_{i=1}^{4}\left(\sum_{j=1}^{m}\int_\Omega K_{ij}(x,t)\widehat{A}^i u(x)dt\right) = f(x), \quad x\in\Omega, \\ D(B_4) &= D(\widehat{A}^4), \end{aligned} \tag{2}$$

where $B_4 : X \to X$ is a linear operator, $\widehat{A}^i$, $i = 1, 2, 3, 4$, are powers, self-compositions, of the operator $\widehat{A}$ and the kernels $K_{ij}(x,t) \in X(\Omega\times\Omega)$ are known functions. We assume that the inverse operator $I = \widehat{A}^{-1}$ is known explicitly, and that the kernels $K_j(x,t)$, $K_{ij}(x,t)$ are separable functions. We establish existence and uniqueness criteria and provide the solutions of the problems (1) and (2) in closed form. Lastly, we investigate under which conditions $B_4 = B^4$ and develop a decomposition technique for obtaining the exact solution to the problem $B^4u(x) = f(x)$.

The outline of the paper is as follows. In Section 1., the two problems are put in a convenient matrix form and exact solution formulae are obtained by a direct method. In Section 2., the conditions for factorizing problem (2) and a decomposition solution method are presented. Finally, two examples are solved in Section 3.

1. The Direct Method

First, we deal with problem (1). We assume that the kernels $K_j(x,t)$ are separable functions, i.e. $K_j(x,t) = g_j(x)h_j(t)$, $j = 1,\ldots,m$, where $g_j(x), h_j(x) \in X$, and introduce the vector of functions

$$\mathbf{g} = \begin{pmatrix} g_1 & \ldots & g_m \end{pmatrix} \in X_m, \quad g_j = g_j(x)\in X,\ j=1,\ldots,m, \tag{3}$$

and the vector of linear and bounded functionals

$$\Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix} \in X_m^*, \quad \Phi_j = \Phi_j(\cdot) = \int_\Omega h_j(t)\cdot dt \in X^*,\ j=1,\ldots,m. \tag{4}$$

Also, by $\Phi(f)$ and $\Phi(\mathbf{g})$ we denote the vector and the $m\times m$ matrix

$$\Phi(f) = \begin{pmatrix} \Phi_1(f) \\ \vdots \\ \Phi_m(f) \end{pmatrix}, \quad \Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1) & \cdots & \Phi_1(g_m) \\ \vdots & \ddots & \vdots \\ \Phi_m(g_1) & \cdots & \Phi_m(g_m) \end{bmatrix}, \tag{5}$$

respectively, and note that

$$\Phi(\mathbf{gN}) = \Phi(\mathbf{g})\mathbf{N}, \tag{6}$$

where $\mathbf{N}$ is an $m\times k$, $k\in\mathbb{N}$, constant matrix. Finally, $\mathbf{1}_m$ symbolizes the $m\times m$ identity matrix and $\mathbf{0}$ the zero column vector.

Thus, the problem (1) can be put in the convenient form

$$Bu = \widehat{A}u - \mathbf{g}\Phi(\widehat{A}u) = f, \quad D(B) = D(\widehat{A}). \tag{7}$$

The conditions for existence and uniqueness, the solution to problem and the correctness of the operator B are provided by the next theorem.

It is recalled here that an operator $P : X \to X$ is said to be *correct* if P is bijective and its inverse P^{-1} is bounded on X.

Theorem 1. *Let X be a complex Banach space, $\widehat{A} : X \to X$ a bijective linear operator and $I = \widehat{A}^{-1}$ its inverse, $\mathbf{g}\in X_m$ and $\Phi \in X_m^*$ as defined in (3) and (4), respectively, and $B : X \to X$ the linear operator*

$$Bu = \widehat{A}u - \mathbf{g}\Phi(\widehat{A}u), \quad D(B) = D(\widehat{A}). \tag{8}$$

Then the following statements are true:

(i) The operator B is bijective on X if and only if

$$\det\mathbf{W} = \det[\mathbf{1}_m - \Phi(\mathbf{g})] \neq 0, \tag{9}$$

and the unique solution to boundary value problem (7), for any $f\in X$, is given by the formula

$$u = B^{-1}f = If + I\mathbf{g}W^{-1}\Phi(f). \tag{10}$$

(ii) If in addition the inverse operator $\widehat{A}^{-1}$ is bounded on X, then the operator B is correct.

Proof. (i) Let $\det \mathbf{W} \neq 0$ and $u \in \ker B$. Then,

$$Bu = \widehat{A}u - \mathbf{g}\Phi(\widehat{A}u) = 0, \tag{11}$$

and after applying the vector Φ on both sides of (11) and utilizing (6),

$$[\mathbf{1}_m - \Phi(\mathbf{g})]\,\Phi(\widehat{A}u) = \mathbf{W}\Phi(\widehat{A}u) = \mathbf{0}, \tag{12}$$

which implies that $\Phi(\widehat{A}u) = \mathbf{0}$. Substitution into (11) yields $Bu = \widehat{A}u = 0$ and hence $u = 0$. This means that $\ker B = \{0\}$ and therefore the operator B is injective. Conversely, we prove that if B is an injective operator then $\det \mathbf{W} \neq 0$, or equivalently, if $\det \mathbf{W} = 0$, then B is not injective. Let $\det \mathbf{W} = 0$. Then there exists a nonzero vector $\mathbf{c} = \mathrm{col}(c_1, \ldots, c_m)$ such that $\mathbf{Wc} = \mathbf{0}$. Consider the element $u_0 = \widehat{A}^{-1}\mathbf{gc}$ and note that $u_0 \neq 0$; otherwise $u_0 = 0$ implies $\mathbf{gc} = 0$ and then $\mathbf{Wc} = [\mathbf{1}_m - \Phi(\mathbf{g})]\mathbf{c} = \mathbf{c} - \Phi(\mathbf{gc}) = \mathbf{c} = \mathbf{0}$ which contradicts the hypothesis that $\mathbf{c}$ is a nonzero vector. From equation (8), we get

$$Bu_0 = \mathbf{gc} - \mathbf{g}\Phi(\mathbf{g})\mathbf{c} = \mathbf{g}[\mathbf{1}_m - \Phi(\mathbf{g})]\mathbf{c} = \mathbf{gWc} = \mathbf{g0} = 0, \tag{13}$$

which means that $\ker B \neq 0$ and so B is not injective.

Further, by multiplying from the left both sides of (7) by $\widehat{A}^{-1}$, we get

$$u - \widehat{A}^{-1}\mathbf{g}\Phi(\widehat{A}u) = \widehat{A}^{-1}f, \tag{14}$$

while acting by the vector Φ on (7), we have

$$[\mathbf{1}_m - \Phi(\mathbf{g})]\,\Phi(\widehat{A}u) = \mathbf{W}\Phi(\widehat{A}u) = \Phi(f). \tag{15}$$

By hypothesis $\det \mathbf{W} \neq 0$ and therefore equation (15) can be solved with respect to $\Phi(\widehat{A}u)$. Substituting $\Phi(\widehat{A}u) = \mathbf{W}^{-1}\Phi(f)$ into (14), we obtain the solution formula (10).

Finally, since the input function f in (7) and (10) is any arbitrary $f \in X$, we have $R(B) = X$ which means that B is bijective.

(ii) Suppose that (9) is true and that the operator $\widehat{A}^{-1}$ is bounded. Then by (i) the operator B is bijective and the unique solution to problem (7) is given by (10). In (10) the operator $\widehat{A}^{-1}$ and the functionals $\Phi_1, \ldots, \Phi_m$ are bounded. This means that the operator B^{-1} is bounded and hence the operator B is correct. □

To find the solution of the boundary value problem (2), we now introduce the vectors of functions

$$\begin{aligned} \mathbf{q} &= \begin{pmatrix} q_1 & \ldots & q_m \end{pmatrix}, \ \mathbf{r} = \begin{pmatrix} r_1 & \ldots & r_m \end{pmatrix}, \ \mathbf{s} = \begin{pmatrix} s_1 & \ldots & s_m \end{pmatrix}, \\ \mathbf{z} &= \begin{pmatrix} z_1 & \ldots & z_m \end{pmatrix}, \ q_j, r_j, s_j, z_j \in X, \ j = 1, \ldots, m, \end{aligned} \tag{16}$$

and write the problem (2) in the form

$$\begin{aligned} B_4 u &= \widehat{A}^4 u - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) = f, \\ D(B_4) &= D(\widehat{A}^4), \end{aligned} \tag{17}$$

and prove the following theorem.

Theorem 2. *Let X be a complex Banach space, $\widehat{A} : X \to X$ a bijective operator and $I = \widehat{A}^{-1}$ its inverse, $\mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z} \in X_m$ and $\Phi \in X_m^*$ as in (16) and (4), respectively, and the operator $B_4 : X \to X$ defined by*

$$\begin{aligned} B_4 u &= \widehat{A}^4 u - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u), \\ D(B_4) &= D(\widehat{A}^4). \end{aligned} \tag{18}$$

Then the following statements are true:

(i) If

$$\det \mathbf{V} \neq 0, \tag{19}$$

where

$$\mathbf{V} = \begin{bmatrix} \Phi(I^3\mathbf{q}) - \mathbf{1}_m & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{z}) \\ \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{r}) - \mathbf{1}_m & \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{z}) \\ \Phi(I\mathbf{q}) & \Phi(I\mathbf{r}) & \Phi(I\mathbf{s}) - \mathbf{1}_m & \Phi(I\mathbf{z}) \\ \Phi(\mathbf{q}) & \Phi(\mathbf{r}) & \Phi(\mathbf{s}) & \Phi(\mathbf{z}) - \mathbf{1}_m \end{bmatrix}, \tag{20}$$

then the operator B_4 is bijective on X and the unique solution to problem (17) is given by

$$\begin{aligned} u &= B_4^{-1} f \\ &= I^4 f - \begin{pmatrix} I^4\mathbf{q} & I^4\mathbf{r} & I^4\mathbf{s} & I^4\mathbf{z} \end{pmatrix} \mathbf{V}^{-1} \begin{pmatrix} \Phi(I^3 f) \\ \Phi(I^2 f) \\ \Phi(I f) \\ \Phi(f) \end{pmatrix}. \end{aligned} \tag{21}$$

(ii) Conversely, if the operator B_4 is injective and also the vectors $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{z}$ *are linearly independent, then* $\det \mathbf{V} \neq 0$.

(iii) If in addition the inverse $I = \widehat{A}^{-1}$ is bounded on X, then the operator B_4 is correct.

Proof. (i) Let $\det \mathbf{V} \neq 0$ and $u \in \ker B_4$, i.e.

$$\widehat{A}^4 u - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) = 0, \tag{22}$$

where $u \in D(\widehat{A}^4)$. We introduce the vectors

$$\bar{\mathbf{q}} = \begin{pmatrix} \mathbf{q} & \mathbf{r} & \mathbf{s} & \mathbf{z} \end{pmatrix}, \quad \bar{\Phi}(u) = \begin{pmatrix} \Phi(\widehat{A}u) \\ \Phi(\widehat{A}^2 u) \\ \Phi(\widehat{A}^3 u) \\ \Phi(\widehat{A}^4 u) \end{pmatrix}, \tag{23}$$

and write (22) conveniently in the matrix form

$$\widehat{A}^4 u - \bar{\mathbf{q}}\bar{\Phi}(u) = 0. \tag{24}$$

By applying the inverse operator $I = \widehat{A}^{-1}$ four times on both sides of (24), we get

$$\widehat{A}^{4-n} u - I^n \bar{\mathbf{q}}\bar{\Phi}(u) = 0, \quad n = 1, 2, 3, \tag{25}$$

$$u - I^4 \bar{\mathbf{q}}\bar{\Phi}(u) = 0. \tag{26}$$

Acting by the vector Φ on both sides of (24) and the three equations (25), we obtain

$$\Phi(\widehat{A}^{4-n} u) - \Phi(I^n \bar{\mathbf{q}})\bar{\Phi}(u) = \mathbf{0}, \quad n = 0, 1, 2, 3, \tag{27}$$

where $I^0 = 1$. Equation (27) can be rearranged in the following configuration

$$\begin{bmatrix} \Phi(I^3\mathbf{q}) - \mathbf{1}_m & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{z}) \\ \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{r}) - \mathbf{1}_m & \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{z}) \\ \Phi(I\mathbf{q}) & \Phi(I\mathbf{r}) & \Phi(I\mathbf{s}) - \mathbf{1}_m & \Phi(I\mathbf{z}) \\ \Phi(\mathbf{q}) & \Phi(\mathbf{r}) & \Phi(\mathbf{s}) & \Phi(\mathbf{z}) - \mathbf{1}_m \end{bmatrix} \bar{\Phi}(u) = \mathbf{0}, \tag{28}$$

or

$$\mathbf{V}\bar{\Phi}(u) = \mathbf{0}, \tag{29}$$

where V is the $4m \times 4m$ matrix in (20). Since $\det \mathbf{V} \neq 0$, it is implied from (29) that $\bar{\Phi}(u) = 0$ and then from (26) that $u = 0$. This means that $\ker B = \{0\}$ and therefore the operator B is injective.

To obtain the solution of (17), we write it in the form

$$\widehat{A}^4 u - \bar{\mathbf{q}}\bar{\Phi}(u) = f, \quad f \in X, \tag{30}$$

By applying the inverse operator $I = \widehat{A}^{-1}$ four times on both sides of (30), we get

$$\widehat{A}^{4-n} u - I^n \bar{\mathbf{q}}\bar{\Phi}(u) = I^n f, \quad n = 1, 2, 3, \tag{31}$$

$$u - I^4 \bar{\mathbf{q}}\bar{\Phi}(u) = I^4 f, \tag{32}$$

and then acting by the vector Φ on both sides of (30) and (31), we have

$$\Phi(\widehat{A}^{4-n} u) - \Phi(I^n \bar{\mathbf{q}})\bar{\Phi}(u) = \Phi(I^n f), \quad n = 0, 1, 2, 3, \tag{33}$$

or in matrix form

$$\mathbf{V}\bar{\Phi}(u) = -\begin{pmatrix} \Phi(I^3 f) \\ \Phi(I^2 f) \\ \Phi(I f) \\ \Phi(f) \end{pmatrix}. \tag{34}$$

By inverting (34) and substituting into (32), we obtain (21). In (17) and (30), f is an arbitrary element of X and therefore $R(B_4) = X$ and so the operator B_4 is bijective and the problem (17) everywhere solvable.

(ii) Let the vectors $\mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z}$ be linearly independent. We will prove that if B_4 is an injective operator then $\det \mathbf{V} \neq 0$, or equivalently, if $\det \mathbf{V} = 0$ then B_4 is not injective. Let $\det \mathbf{V} = 0$. Then there exists a nonzero vector $\mathbf{c} = \mathrm{col}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4)$ such that $\mathbf{V}\mathbf{c} = \mathbf{0}$ where $\mathbf{c}_i = col\,(c_{i1}, ..., c_{im})$, $i = 1, 2, 3, 4$. Consider the element $u_0 = I^4(\mathbf{q}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{s}\mathbf{c}_3 + \mathbf{z}\mathbf{c}_4) \in D(\widehat{A}^4)$.

Substituting this element into (18), we obain

$$\begin{aligned} Bu_0 &= \mathbf{q}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{s}\mathbf{c}_3 + \mathbf{z}\mathbf{c}_4 \\ &\quad -\mathbf{q}\left[\Phi(I^3\mathbf{q})\mathbf{c}_1 + \Phi(I^3\mathbf{r})\mathbf{c}_2 + \Phi(I^3\mathbf{s})\mathbf{c}_3 + \Phi(I^3\mathbf{z})\mathbf{c}_4\right] \\ &\quad -\mathbf{r}\left[\Phi(I^2\mathbf{q})\mathbf{c}_1 + \Phi(I^2\mathbf{r})\mathbf{c}_2 + \Phi(I^2\mathbf{s})\mathbf{c}_3 + \Phi(I^2\mathbf{z})\mathbf{c}_4\right] \\ &\quad -\mathbf{s}\left[\Phi(I\mathbf{q})\mathbf{c}_1 + \Phi(I\mathbf{r})\mathbf{c}_2 + \Phi(I\mathbf{s})\mathbf{c}_3 + \Phi(I\mathbf{z})\mathbf{c}_4\right] \\ &\quad -\mathbf{z}\left[\Phi(\mathbf{q})\mathbf{c}_1 + \Phi(\mathbf{r})\mathbf{c}_2 + \Phi(\mathbf{s})\mathbf{c}_3 + \Phi(\mathbf{z})\mathbf{c}_4\right] \\ &= -\begin{pmatrix} \mathbf{q} & \mathbf{r} & \mathbf{s} & \mathbf{z} \end{pmatrix} \mathbf{V}\mathbf{c} = 0. \end{aligned} \tag{35}$$

This means that $u_0 \in \ker B_4$. Note that $u_0 \neq 0$, because by hypothesis $\mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z}$ are linearly independent and $\mathbf{c} \neq \mathbf{0}$. Hence, B_4 is not injective.

(iii) Let additionally $I = \widehat{A}^{-1}$ is bounded, i.e. $\widehat{A}$ is correct. Then the operators I^i, $i = 2, 3, 4$, are bounded and since the functionals Φ_j, $j = 1, \ldots, m$, are bounded on X, it is concluded that the operator B_4^{-1} is bounded too by means of (21). Hence, if the operator $\widehat{A}$ is correct and the condition (19) is satisfied then the operator B_4 is correct. □

2. Decomposition Method

Let the operator $B : X \to X$ be defined by

$$Bu = \widehat{A}u - \mathbf{z}\Phi(\widehat{A}u), \quad D(B) = D(\widehat{A}), \tag{36}$$

where $\mathbf{z} \in D(\widehat{A}^3)_m$, and let the vectors

$$\mathbf{q} = \widehat{A}\mathbf{r} - \mathbf{z}\Phi(\widehat{A}\mathbf{r}), \quad \mathbf{r} = \widehat{A}\mathbf{s} - \mathbf{z}\Phi(\widehat{A}\mathbf{s}), \quad \mathbf{s} = \widehat{A}\mathbf{z} - \mathbf{z}\Phi(\widehat{A}\mathbf{z}). \tag{37}$$

In this case, problem (2) can be formulated as

$$B_4 u = B^4 u = f, \quad D(B_4) = D(\widehat{A}^4). \tag{38}$$

It is recognized that conditions (37) are seldom met in real life problems, but when it happens the solution process is simplified further. Moreover, the results derived here are of theoretical importance and valuable for extra developments.

Theorem 3. *Let X be a complex Banach space, $\widehat{A} : X \to X$ a bijective operator and $I = \widehat{A}^{-1}$ its inverse and $\Phi \in X_m^*$ as in (4). Let $\mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z} \in X_m$ satisfy (37) and the operator $B : X \to X$ be defined by (36). Then the following statements are true:*

(i) The operator

$$\begin{aligned} B_4 u &= \widehat{A}^4 u - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u), \\ D(B_4) &= D(\widehat{A}^4), \end{aligned} \tag{39}$$

is biquadratic, meaning $B_4 u = B^4 u$.

(ii) If

$$\det \mathbf{W} = \det[\mathbf{1}_m - \Phi(\mathbf{z})] \neq 0, \tag{40}$$

then the operator B_4 is bijective and the unique solution to boundary value problem (38) is obtained by

$$\begin{aligned} u &= B_4^{-1} f = u_4, \\ u_{k+1} &= I u_k + I\mathbf{z}\mathbf{W}^{-1}\Phi(u_k), \quad k = 0, 1, 2, 3, \quad u_0 = f. \end{aligned} \tag{41}$$

(iii) Conversely, if the operator B_4 is injective and the components of the vector $\mathbf{z}$ are linearly independent, then $\det \mathbf{W} \neq 0$.

(iv) Finally, if (40) holds true and in addition the inverse operator $I = \widehat{A}^{-1}$ is bounded on X, then B_4 is correct.

Proof. (i) From (37) and (36), we have

$$\begin{aligned} \mathbf{s} &= B\mathbf{z}, \\ \mathbf{r} &= B\mathbf{s} = B^2\mathbf{z}, \\ \mathbf{q} &= B\mathbf{r} = B^3\mathbf{z}, \end{aligned} \tag{42}$$

where the operators $B^2, B^3 : X \to X$ with $D(B^2) = D(\widehat{A}^2)$ and $D(B^3) = D(\widehat{A}^3)$. Substituting (42) into (39), we get

$$\begin{aligned} B_4u &= \widehat{A}^4u - B^3\mathbf{z}\Phi(\widehat{A}u) - B^2\mathbf{z}\Phi(\widehat{A}^2u) - B\mathbf{z}\Phi(\widehat{A}^3u) - \mathbf{z}\Phi(\widehat{A}^4u) \\ &= B\widehat{A}^3u - B\left[B^2\mathbf{z}\Phi(\widehat{A}u) + B\mathbf{z}\Phi(\widehat{A}^2u) + \mathbf{z}\Phi(\widehat{A}^3u)\right] \\ &= B\left[\widehat{A}^3u - B^2\mathbf{z}\Phi(\widehat{A}u) - B\mathbf{z}\Phi(\widehat{A}^2u) - \mathbf{z}\Phi(\widehat{A}^3u)\right] \\ &= B\left[B\widehat{A}^2u - B^2\mathbf{z}\Phi(\widehat{A}u) - B\mathbf{z}\Phi(\widehat{A}^2u)\right] \\ &= B^2\left[\widehat{A}^2u - B\mathbf{z}\Phi(\widehat{A}u) - \mathbf{z}\Phi(\widehat{A}^2u)\right] \\ &= B^2\left[B\widehat{A}u - B\mathbf{z}\Phi(\widehat{A}u)\right] \\ &= B^3\left[\widehat{A}u - \mathbf{z}\Phi(\widehat{A}u)\right] \\ &= B^4u. \end{aligned} \tag{43}$$

(ii) Let $\det\mathbf{W} \neq 0$. Then by Theorem 1 the operator B in (36) is bijective and consequently the operator $B_4 = B^4$ is again bijective as composition of bijective linear operators. Thus, the boundary value problem (38) may be solved as follows

$$\begin{aligned} B^4u &= u_0, \quad u_0 = f, \\ B^3u &= B^{-1}u_0 = u_1, \quad u_1 = Iu_0 + I\mathbf{z}\mathbf{W}^{-1}\Phi(u_0), \\ B^2u &= B^{-1}u_1 = u_2, \quad u_2 = Iu_1 + I\mathbf{z}\mathbf{W}^{-1}\Phi(u_1), \\ Bu &= B^{-1}u_2 = u_3, \quad u_3 = Iu_2 + I\mathbf{z}\mathbf{W}^{-1}\Phi(u_2), \\ u &= B^{-1}u_3 = u_4, \quad u_4 = Iu_3 + I\mathbf{z}\mathbf{W}^{-1}\Phi(u_3), \end{aligned} \tag{44}$$

which yields (41).

(iii) Suppose that the components of the vector $\mathbf{z}$ are linearly independent and we will prove that if B_4 is an injective operator then $\det\mathbf{W} \neq 0$, or equivalently, if $\det\mathbf{W} = 0$ then B_4 is not injective. Let $\det\mathbf{W} = 0$. Then there exists a nonzero vector $\mathbf{c} = \mathrm{col}(c_1, \ldots, c_m)$ such that $\mathbf{Wc} = \mathbf{0}$. Consider the element $u_0 = I\mathbf{zc}$ and notice that $u_0 \neq 0$ because the components of the vector $\mathbf{z}$ are linearly independent and $\mathbf{c}$ is nonzero. From (39) and (42), we obtain

$$\begin{aligned} B_4u_0 &= \widehat{A}^4u_0 - B^3\mathbf{z}\Phi(\widehat{A}u_0) - B^2\mathbf{z}\Phi(\widehat{A}^2u_0) - B\mathbf{z}\Phi(\widehat{A}^3u_0) - \mathbf{z}\Phi(\widehat{A}^4u_0) \\ &= \widehat{A}^3\mathbf{zc} - B^3\mathbf{z}\Phi(\mathbf{z})\mathbf{c} - B^2\mathbf{z}\Phi(\widehat{A}\mathbf{z})\mathbf{c} - B\mathbf{z}\Phi(\widehat{A}^2\mathbf{z})\mathbf{c} - \mathbf{z}\Phi(\widehat{A}^3\mathbf{z})\mathbf{c} \\ &= \left[B\widehat{A}^2\mathbf{z} - B^3\mathbf{z}\Phi(\mathbf{z}) - B^2\mathbf{z}\Phi(\widehat{A}\mathbf{z}) - B\mathbf{z}\Phi(\widehat{A}^2\mathbf{z})\right]\mathbf{c} \\ &= B\left[\widehat{A}^2\mathbf{z} - B^2\mathbf{z}\Phi(\mathbf{z}) - B\mathbf{z}\Phi(\widehat{A}\mathbf{z}) - \mathbf{z}\Phi(\widehat{A}^2\mathbf{z})\right]\mathbf{c} \\ &= B\left[B\widehat{A}\mathbf{z} - B^2\mathbf{z}\Phi(\mathbf{z}) - B\mathbf{z}\Phi(\widehat{A}\mathbf{z})\right]\mathbf{c} \\ &= B^2\left[\widehat{A}\mathbf{z} - B\mathbf{z}\Phi(\mathbf{z}) - \mathbf{z}\Phi(\widehat{A}\mathbf{z})\right]\mathbf{c} \\ &= B^2\left[B\mathbf{z} - B\mathbf{z}\Phi(\mathbf{z})\right]\mathbf{c} \\ &= B^3\mathbf{z}\left[\mathbf{1}_m - \Phi(\mathbf{z})\right]\mathbf{c} \\ &= B^3\mathbf{zWc} = 0. \end{aligned} \tag{45}$$

From (45) it is implied that $u_0 \in \ker B_4$ and hence B_4 is not injective.

(iv) Assume (40) holds true. Then the operator B_4 is bijective and its inverse is given by (41). If in addition $\widehat{A}^{-1}$ is bounded and since the functionals $\Phi_1, \ldots, \Phi_m$ in (41) are bounded, it is implied that the operator B_4^{-1} is bounded and hence B_4 is correct. □

3. Examples

In this section, we select and solve two representative example problems to explain the implementation as well as to show the efficiency of the techniques presented in the previous sections.

Example 1. Let $\Omega = \left\{x \in \mathbb{R}^3 : |x| < 1\right\}$ and $\partial\Omega = \left\{x \in \mathbb{R}^3 : |x| = 1\right\}$. Consider the boundary value problem

$$\begin{gathered} \triangle u(x) - \textstyle\sum_{j=1}^{2} g_j(x)\int_\Omega v_j(y)\triangle u(y)dy = f(x), \quad x \in \Omega, \\ u_{|\partial\Omega} = 0, \end{gathered} \tag{46}$$

where $\triangle$ is the Laplace operator in $\mathbb{R}^3$. We take $X = L_2(\Omega)$ and

$$\begin{aligned} \widehat{A}u(x) &= \triangle u(x), \quad D(\widehat{A}) = \{u(x) \in \mathbb{W}_2^2(\Omega) : u_{|\partial\Omega} = 0\}, \\ \Phi_i(\cdot) &= \int_\Omega v_i(x) \cdot dx, \quad i = 1, 2, \\ \mathbf{g} &= \begin{pmatrix} g_1(x) & g_2(x) \end{pmatrix}, \end{aligned} \tag{47}$$

and thus the problem (46) is put in the form

$$Bu(x) = \widehat{A}u(x) - \mathbf{g}\Phi(\widehat{A}u(x)) = f(x), \quad D(B) = D(\widehat{A}). \tag{48}$$

It is known that the problem

$$\widehat{A}u(x) = \triangle u(x) = f(x), \quad u(x)_{|\partial\Omega} = 0, \quad u \in \mathbb{W}_2^2(\Omega), \quad f(x) \in L_2(\Omega). \tag{49}$$

is uniquely and everywhere solvable on $L_2(\Omega)$ and

$$u(x) = \widehat{A}^{-1}f(x) = \int_\Omega \mathcal{G}(x,y)f(y)dy, \quad \forall f \in L_2(\Omega), \tag{50}$$

where $\mathcal{G}(x,y) = \mathcal{G}_1(x,y) + g(x,y)$ is the Green's function, $\mathcal{G}_1(x,y) = \frac{1}{4\pi|x-y|}$ and $g(x,y)$ is a harmonic function in Ω and belongs to $C^\infty(\bar{\Omega})$. Also, the operator $\widehat{A}^{-1}$ is bounded and hence $\widehat{A}$ is correct. We compute,

$$\det W = \det\left[\mathbf{1}_2 - \Phi(\mathbf{g})\right] = \det \begin{bmatrix} 1 - \Phi_1(g_1(x)) & -\Phi_1(g_2(x)) \\ -\Phi_2(g_1(x)) & 1 - \Phi_2(g_2(x)) \end{bmatrix}. \tag{51}$$

If $\det W \neq 0$, then the problem (46) admits a unique solution which is obtained by substituting into (10). Finally, the operator B by Theorem 1 is correct.

Example 2. Consider the fourth order integro-differential equation

$$\begin{aligned} u^{(4)}(t) \quad & + \frac{3}{64}(5101t^3 - 32978t^2 + 102042t - 148458) \int_0^1 xu'(x)dx \\ & - \frac{1}{16}(593t^3 - 3834t^2 + 11874t - 17266) \int_0^1 xu''(x)dx \\ & + \frac{1}{4}(23t^3 - 150t^2 + 462t - 670) \int_0^1 xu'''(x)dx \\ & - (t^3 - 6t^2 + 18t - 26) \int_0^1 xu^{(4)}(x)dx \\ = & - \frac{279}{640}(5101t^3 - 32978t^2 + 102042t - 148458), \end{aligned} \tag{52}$$

subject to boundary conditions

$$u^{(i)}(0) = 2u^{(i)}(1), \quad i = 0, 1, 2, 3. \tag{53}$$

We take $X = C[0,1]$,

$$\widehat{A}u = u'(t), \quad D(\widehat{A}) = \{u(t) \in C^1[0,1] : u(0) = 2u(1)\}, \tag{54}$$

consequently,

$$\begin{aligned} \widehat{A}^2 u = u''(t), \quad & D(\widehat{A}^2) = \left\{u \in C^2[0,1] : u^{(i)}(0) = 2u^{(i)}(1),\ i = \overline{0,1}\right\}, \\ \widehat{A}^3 u = u'''(t), \quad & D(\widehat{A}^3) = \left\{u \in C^3[0,1] : u^{(i)}(0) = 2u^{(i)}(1),\ i = \overline{0,2}\right\}, \\ \widehat{A}^4 u = u^{(4)}(t), \ & D(\widehat{A}^4) = \left\{u \in C^4[0,1] : u^{(i)}(0) = 2u^{(i)}(1),\ i = \overline{0,3}\right\}, \end{aligned} \tag{55}$$

and

$$\begin{aligned} \Phi(\widehat{A}^i u) &= \left(\int_0^1 xu^{(i)}(x)dx\right), \quad i = 1, 2, 3, 4, \\ \mathbf{q} &= \left(-\frac{3}{64}\left(5101t^3 - 32978t^2 + 102042t - 148458\right)\right), \\ \mathbf{r} &= \left(\frac{1}{16}\left(593t^3 - 3834t^2 + 11874t - 17266\right)\right), \\ \mathbf{s} &= \left(-\frac{1}{4}\left(23t^3 - 150t^2 + 462t - 670\right)\right), \\ \mathbf{z} &= \left(t^3 - 6t^2 + 18t - 26\right), \\ f &= -\frac{279}{640}(5101t^3 - 32978^2 + 102042t - 148458). \end{aligned} \tag{56}$$

It is easy to verify that the operator $\widehat{A}$ is correct and its inverse is given by

$$\widehat{A}^{-1}f(t) = \int_0^t f(x)dx - 2\int_0^1 f(x)dx, \quad f \in C[0,1]. \tag{57}$$

and that the functional Φ is bounded, i.e. $\Phi \in C[0,1]^* = X^*$. Thus, the boundary value problem (52), (53) can be expressed now in the operator form (17).

To find its solution we can apply Theorem 2. However, by inspecting the operator $\widehat{A}$, the vector of functionals Φ and the vectors $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{z}$, we can verify that the conditions (37) are satisfied, and therefore the decomposition Theorem 3 is more appropriate, which is much easier to implement. It is straightforward to show that

$$\begin{aligned} \det \mathbf{W} &= \det\left[1 - \Phi(\mathbf{z})\right] \\ &= \left[1 - \int_0^1 x(x^3 - 6x^2 + 18x - 26)dx\right] = \frac{93}{10} \neq 0. \end{aligned} \tag{58}$$

Thus, the problem (52), (53) is correct and it can be formulated as $B^4u = f$, where B is the first order operator in (36). By substituting into (41), we obtain the unique solution of the problem (52), (53), which is

$$u(t) = \frac{t^4}{4} - 2t^3 + 9t^2 - 26t + \frac{75}{2}. \tag{59}$$

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Научная статья

DOI: 10.18287/2541-7525-2020-26-1-14-22

УДК 629.7.05

Дата: поступления статьи: 18.12.2019
после рецензирования: 20.01.2020
принятия статьи: 28.02.2020

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О РЕШЕНИИ НЕКОТОРЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВЫСШЕГО ПОРЯДКА СПЕЦИАЛЬНОГО ВИДА

АННОТАЦИЯ

Статья посвящена решению краевых задач для линейных интегро-дифференциальных уравнений высшего порядка типа Фредгольма с дифференциальными и интегральными операторами, охватывающими степени идеального биективного линейного дифференциального оператора, обратный которому явно известен. Выводятся условия существования и единственности решений, и решения изложены в закрытом виде. Подход основан на представлении о том, что интегро-дифференциальный оператор является возмущенным дифференциальным оператором. Полученные результаты имеют как теоретическое, так и практическое значение. Метод поясняется решением двух иллюстративных задач.

Ключевые слова: интегро-дифференциальные уравнения, задачи с начальными условиями, граничные задачи, дифференциальные операторы, энергетические операторы, смешанные продукты, точные решения.

Цитирование. Providas E., Parasidis I.N. On the solution of some higher-order integro-differential equations of special form // Вестник Самарского университета. Естественнонаучная серия. 2020. Т. 26, № 1. С. 14–22. DOI: http://doi.org/10.18287/ 2541-7525-2020-26-1-14-22.

Информация о конфликте интересов: авторы и рецензенты заявляют об отсутствии конфликта интересов.

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