УДК 517.968.7 Дата поступления статьи: $4/X/2019$
DOI: 10.18287/2541-7525-2019-25-4-14-21 Дата принятия статьи: $18/X/2019$

М.М. Байбурин

ОБОБЩЕНИЯ ДЛЯ НЕКОТОРЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ, СОДЕРЖАЩИХ СТЕПЕНИ ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА

© *Байбурин Мерхасыл Мукеевич* — кандидат физико-математических наук, доцент кафедры фундаментальной математики, Евразийский национальный университет им. Л.Н. Гумилева, 010008, Республика Казахстан, г. Нур-Султан, ул. Сатпаева, 2.
E-mail: merkhasyl@mail.ru. ORCID: https://orcid.org/0000-0003-0083-6142

АННОТАЦИЯ

В данной статье исследуются абстрактные уравнения, содержащие операторы второй, третьей и четвертой степени.

Необходимые условия разрешимости для абстрактных уравнений, содержащих операторы второй и четвертой степени, доказаны без применения линейной независимости векторов, входящих в данные уравнения. Некоторые авторы существенно использовали линейную независимость векторов для доказательства необходимого условия разрешимости.

В данной статье также дается критерий корректности для абстрактного уравнения, содержащего операторы третьей степени с произвольными векторами, и его решение в терминах этих операторов в банаховом пространстве.

Теория, представленная здесь, может быть полезна для исследования интегро-дифференциальных уравнений Фредгольма, содержащих степени некоторого обыкновенного дифференциального оператора или дифференциального оператора в частных производных.

Ключевые слова: интегро-дифференциальные уравнения Фредгольма, начальные задачи, краевые задачи, дифференциальные операторы, степенные операторы, точные решения.

Цитирование. Baiburin M.M. Generalizations to some Integro-differential equations embodying powers of a differential operator // Вестник Самарского университета. Естественнонаучная серия. 2019. Т. 25. № 4. С. 14–21. DOI: http://doi.org/10.18287/2541-7525-2019-25-4-14-21.

UDC 517.968.7
DOI: 10.18287/2541-7525-2019-25-4-14-21

Submitted: $4/X/2019$
Accepted: $18/X/2019$

M.M. Baiburin

GENERALIZATIONS TO SOME INTEGRO-DIFFERENTIAL EQUATIONS EMBODYING POWERS OF A DIFFERENTIAL OPERATOR

© *Baiburin Merkhassyl Mokeevich* — Candidate of Physical and Mathematical Sciences, associate professor of the Department of Fundamental Mathematics, L.N. Gumilyov Eurasian National University, 2, Satpayev street, Nur-Sultan, 010008, Republic of Kazakhstan.
E-mail: merkhasyl@mail.ru. ORCID: https://orcid.org/0000-0003-0083-6142

ABSTRACT

The abstract equations containing the operators of the second, third and fourth degree are investigated in this work.

The necessary conditions for the solvability of the abstract equations, containing the operators of the second and fourth degree, are proved without using linear independence of the vectors included in these equations. Previous authors have essentially used the linear independence of the vectors to prove the necessary solvability condition.

The present paper also gives the correctness criterion for the abstract equation, containing the operators of the third degree with arbitrary vectors, and its exact solution in terms of these vectors in a Banach space.

The theory presented here, can be useful for investigation of Fredholm integro-differential equations embodying powers of an ordinary differential operator or a partial differential operator.

Key words: fredholm Integro-differential equations, initial value problems, boundary value problems, differential operators, power operators, composite products, exact solutions.

Citation. Baiburin M.M. *Obobshcheniya dlya nekotorykh integro-differentsial'nykh uravnenii, soderzhashchikh stepeni differentsial'nogo operatora* [Generalizations to some integro-differential equations embodying powers of a differential operator]. *Vestnik Samarskogo universiteta. Estestvennonauchnaya seriya* [Vestnik of Samara University. Natural Science Series], 2019, no. 25, no. 4, pp. 14–21. DOI: http://doi.org/10.18287/2541-7525-2018-25-4-14-21 [in Russian].

Introduction

Boundary value problems (BVPs) for integro-differential equations (IDEs) with initial and nonlocal boundary conditions arise in various fields of mechanics, physics, biology, biotechnology, chemical engineering, medical science, finance and others. (see [1; 3–5; 12; 14].) Finding an exact solution of BVPs for Fredholm IDEs is a difficult problem and is given in [2; 6–11; 13; 15–17]. IDEs embodying powers of a differential operator of the type

$$\begin{aligned} &B_2u = \widehat{A}^2u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2u) = f,\\ &D(B_2) = D(\widehat{A}^2), \end{aligned} \tag{1}$$

$$\begin{aligned} &B_3u = \widehat{A}^3u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2u) - \mathbf{s}\Phi(\widehat{A}^3u) = f,\\ &D(B_3) = D(\widehat{A}^3), \end{aligned} \tag{2}$$

$$\begin{aligned} &B_4u = \widehat{A}^4u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2u) - \mathbf{s}\Phi(\widehat{A}^3u) - \mathbf{z}\Phi(\widehat{A}^4u) = f,\\ &D(B_4) = D(\widehat{A}^4). \end{aligned} \tag{3}$$

are solved usually by more simple methods. Such equations are used for solving Fredholm ordinary and partial IDEs with initial and boundary conditions, when $\widehat{A}$ is a differential operator, functionals $\Phi(u)$, $\Phi(\widehat{A}^iu)$, $i = 1, 2, 3, 4$ are Fredholm integral operators with separable kernels. For example, the problem [13]
$z''(x) - \lambda \int_0^1 x[z''(y) + z(y)]dy = x, \quad z(0) = 1, \quad z'(0) = 0,$
by substitution $u(x) = z(x) - 1$ is reduced to
$u''(x) - \lambda x \int_0^1 [u''(y) + u(y)]dy = (1 + 2\lambda)x, \quad u(0) = 0, \quad u'(0) = 0,$
which is of the type (1), where
$D(\widehat{A}) = \{u(x) \in C^1[0,1] : u(0) = 0\}$, $D(\widehat{A}^2) = \{u(x) \in C^2[0,1] : u(0) = u'(0) = 0\}$, $\mathbf{p} = \mathbf{r} = \lambda x, \mathbf{q} = \mathbf{0}$, $\Phi(u) = \int_0^1 u(y)dy$, $\Phi(\widehat{A}^2u) = \int_0^1 u''(y)dy$.

The operator B_3 with the third degree operator $\widehat{A}$ which we study in (2), has a more complex shape then in [9]. The main result of this paper is Theorem 1.3. Finally, we give one example of integro-differential equation demonstrating the power and usefulness of the method presented. By C we denote the set of all complex numbers and by X, Y the complex Banach spaces. The domain and range of a linear operator $P : X \to Y$ will be designated by $D(A)$ and $R(A)$, respectively. We recall that a linear operator $P : X \to Y$ is said to be *injective (or uniquely solvable)* if for all $u_1, u_2 \in D(P)$ such that $Pu_1 = Pu_2$, follows that $u_1 = u_2$, alternatively, the operator P is *injective* if and only if $\ker P = \{0\}$. A linear operator $P : X \to Y$ is called *surjective (or everywhere solvable)* if $R(P) = Y$. The operator P is called *bijective* if P is both injective and surjective. Lastly, P is said to be correct if P is bijective and its inverse P^{-1} is bounded on Y. If an operator P is injective (correct), then the corresponding *equation* $Pu = f$ is called *uniquely solvable (correct)*.

If $\Psi_i \in X^*, i = 1, \ldots, m,$ then we denote by $\Psi = col(\Psi_1, \ldots, \Psi_m)$ and $\Psi(x) = \mathrm{col}(\Psi_1(x), \ldots, \Psi_m(x))$. If $g_1, \ldots, g_m \in X$, then we write $g = (g_1, \ldots, g_m) \in X_m$. We will denote by $\Psi(g)$ the $n \times n$ matrix whose i, j-th entry $\Psi_i(g_j)$ is the value of functional Ψ_i on element g_j. It is easy to verify that for a constant $m \times k$ matrix C holds $\Psi(gC) = \Psi(g)C$. We denote below by 0_m the zero matrix and by I_m the identity $m \times m$ matrix. By $\mathbf{0}$ we will denote the zero column vector.

1. Main Results

First we generalize Theorem 1 [7], where prove the necessary solvability condition of the operator B_2 without to clame the linear independence of the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$.

Theorem 1.1 Let X be a complex Banach space, $\widehat{A}$ a bijective linear operator and $I = \widehat{A}^{-1}$ its inverse, $\Phi = col(\Phi_1, ..., \Phi_m) \in X_m^*$, and $\mathbf{p} = (p_1, ..., p_m)$, $\mathbf{q} = (q_1, ..., q_m)$, $\mathbf{r} = (r_1, ..., r_m) \in X_m$. Let the operator $B_2 : X \to X$ be defined by

$$\begin{aligned} B_2 u &= \widehat{A}^2 u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) = f, \\ D(B_2) &= D(\widehat{A}^2). \end{aligned} \tag{4}$$

The following statements are true:

(i) The operator B_2 is injective (uniquely solvable) if and only if

$$\det \mathbf{W} = \det \begin{bmatrix} \Phi(\mathbf{r}) - \mathbf{1}_m & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) - \mathbf{1}_m & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{r}) & \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) - \mathbf{1}_m \end{bmatrix} \neq 0. \tag{5}$$

(ii) If the operator B_2 is injective, then it is bijective and the unique solution to (4) for any $f \in X$ is given by

$$u = B_2^{-1} f = I^2 f - \begin{pmatrix} I^2\mathbf{r} & I^2\mathbf{q} & I^2\mathbf{p} \end{pmatrix} \mathbf{W}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2 f) \end{pmatrix}. \tag{6}$$

(iii) If the inverse operator $I = \widehat{A}^{-1}$ is bounded on X, that is, $\widehat{A}$ is correct, then the operator B_2 is correct.

Proof. (i) The sufficient solvability condition is proved as in [7]. We prove now the neccesary solvability condition, i.e. we prove that if B_2 is an injective operator then $\det W \neq 0$, or equivalently, if $\det W = 0$, then B_2 is not injective. Let $\det W = 0$. Then there exists a nonzero vector of constants $\mathbf{c} = col(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$, where $\mathbf{c}_i = col(c_{i1}, ..., c_{im})$, $i = 1, 2, 3$, such that $W\mathbf{c} = \mathbf{0}$. Consider the element $u_0 = I^2(\mathbf{r}\mathbf{c}_1 + \mathbf{q}\mathbf{c}_2 + \mathbf{p}\mathbf{c}_3) \in D(\widehat{A}^2)$. Note that $u_0 \neq 0$, alternatively $\mathbf{r}\mathbf{c}_1 + \mathbf{q}\mathbf{c}_2 + \mathbf{p}\mathbf{c}_3 = 0$ and from $W\mathbf{c} = \mathbf{0}$ and the linearity of a functional vector Φ follows that

$$\begin{aligned} W\mathbf{c} &= \begin{bmatrix} \Phi(\mathbf{r}) - \mathbf{1}_m & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) - \mathbf{1}_m & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{r}) & \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) - \mathbf{1}_m \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} \\ &= \begin{bmatrix} \Phi(\mathbf{r})\mathbf{c}_1 - \mathbf{c}_1 + \Phi(\mathbf{q})\mathbf{c}_2 + \Phi(\mathbf{p})\mathbf{c}_3 \\ \Phi(I\mathbf{r})\mathbf{c}_1 + \Phi(I\mathbf{q})\mathbf{c}_2 - \mathbf{c}_2 + \Phi(I\mathbf{p})\mathbf{c}_3 \\ \Phi(I^2\mathbf{r})\mathbf{c}_1 + \Phi(I^2\mathbf{q})\mathbf{c}_2 + \Phi(I^2\mathbf{p})\mathbf{c}_3 - \mathbf{c}_3 \end{bmatrix} = \\ &= \begin{bmatrix} \Phi(\mathbf{r}\mathbf{c}_1 + \mathbf{q}\mathbf{c}_2 + \mathbf{p}\mathbf{c}_3) - \mathbf{c}_1 \\ \Phi\left(I(\mathbf{r}\mathbf{c}_1 + \mathbf{q}\mathbf{c}_2 + \mathbf{p}\mathbf{c}_3)\right) - \mathbf{c}_2 \\ \Phi\left(I^2(\mathbf{r}\mathbf{c}_1 + \mathbf{q}\mathbf{c}_2 + \mathbf{p}\mathbf{c}_3)\right) - \mathbf{c}_3 \end{bmatrix} = - \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \end{aligned}$$

Then $\mathbf{c}_i = \mathbf{0}$, $i = 1, 2, 3$ and so $\mathbf{c} = \mathbf{0}$. But by hypothesis $\mathbf{c} \neq \mathbf{0}$. So $u_0 \neq \mathbf{0}$. That $u_0 \in \ker B_2$ is proved as in [7]. Thus we proved that, if B_2 is an injective operator then $\det W \neq 0$. Statments (ii), (iii) are proved as in Theorem 1 [7].

We generalize Theorem 2 [7], where prove the necessary solvability condition of the operator B_4 without to clame the linear independence of the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z}$.

Theorem 1.2 Let the space X, the operator $\widehat{A}$ and its inverse operator $I = \widehat{A}^{-1}$, and the vectors Φ, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, as above. Let $\mathbf{s} = (s_1, ..., s_m)$, $\mathbf{z} = (z_1, ..., z_m) \in X_m$ and the operator $B_4 : X \to X$ be defined by

$$\begin{aligned} B_4 u &= \widehat{A}^4 u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) = f, \\ D(B_4) &= D(\widehat{A}^4), \end{aligned} \tag{7}$$

where $f \in X$. Then the following statements are true:

(i) The operator B_4 is injective if and only if

$$\det \mathbf{V} \neq 0,$$

where

$$\mathbf{V} = \begin{bmatrix} \Phi(\mathbf{z}) - \mathbf{1}_m & \Phi(\mathbf{s}) & \Phi(\mathbf{r}) & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{z}) & \Phi(I\mathbf{s}) - \mathbf{1}_m & \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{z}) & \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) - \mathbf{1}_m & \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) \\ \Phi(I^3\mathbf{z}) & \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{q}) - \mathbf{1}_m & \Phi(I^3\mathbf{p}) \\ \Phi(I^4\mathbf{z}) & \Phi(I^4\mathbf{s}) & \Phi(I^4\mathbf{r}) & \Phi(I^4\mathbf{q}) & \Phi(I^4\mathbf{p}) - \mathbf{1}_m \end{bmatrix}.$$

(ii) If the operator B_4 is injective, then it is bijective and the unique solution to (7) for any $f \in X$ is given by

$$u = I^4 f - \begin{pmatrix} I^4\mathbf{z} & I^4\mathbf{s} & I^4\mathbf{r} & I^4\mathbf{q} & I^4\mathbf{p} \end{pmatrix} \mathbf{V}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2 f) \\ \Phi(I^3 f) \\ \Phi(I^4 f) \end{pmatrix}. \tag{8}$$

(iii) If the inverse operator $I = \widehat{A}^{-1}$ is bounded on X, that is, $\widehat{A}$ is correct, then the operator B_4 is correct.

Proof. (i) The sufficient solvability condition is proved as in [7]. We prove now the neccesary solvability condition, i.e. we prove that if B_4 is an injective operator then $\det \mathbf{V} \neq 0$, or equivalently, if $\det \mathbf{V} = 0$, then B_4 is not injective. Let $\det \mathbf{V} = 0$. Then there exists a nonzero vector of constants $\mathbf{c} = col(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5)$, where $\mathbf{c}_i = col(c_{i1}, ..., c_{im})$, $i = 1, ..., 5$ such that $V\mathbf{c} = \mathbf{0}$. Consider the element $u_0 = I^4(\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5) \in D(\widehat{A}^4)$. Note that $u_0 \neq 0$, alternatively $\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5 = 0$ and from $V\mathbf{c} = \mathbf{0}$ and the linearity of a functional vector Φ follows that

$$\begin{aligned} \mathbf{Vc} &= \begin{bmatrix} \Phi(\mathbf{z}) - \mathbf{1}_m & \Phi(\mathbf{s}) & \Phi(\mathbf{r}) & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{z}) & \Phi(I\mathbf{s}) - \mathbf{1}_m & \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{z}) & \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) - \mathbf{1}_m & \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) \\ \Phi(I^3\mathbf{z}) & \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{q}) - \mathbf{1}_m & \Phi(I^3\mathbf{p}) \\ \Phi(I^4\mathbf{z}) & \Phi(I^4\mathbf{s}) & \Phi(I^4\mathbf{r}) & \Phi(I^4\mathbf{q}) & \Phi(I^4\mathbf{p}) - \mathbf{1}_m \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{c}_4 \\ \mathbf{c}_5 \end{bmatrix} \\ &= \begin{bmatrix} \Phi(\mathbf{z})\mathbf{c}_1 - \mathbf{c}_1 + \Phi(\mathbf{s})\mathbf{c}_2 + \Phi(\mathbf{r})\mathbf{c}_3 + \Phi(\mathbf{q})\mathbf{c}_4 + \Phi(\mathbf{p})\mathbf{c}_5 \\ \Phi(I\mathbf{z})\mathbf{c}_1 + \Phi(I\mathbf{s})\mathbf{c}_2 - \mathbf{c}_2 + \Phi(I\mathbf{r})\mathbf{c}_3 + \Phi(I\mathbf{q})\mathbf{c}_4 + \Phi(I\mathbf{p})\mathbf{c}_5 \\ \Phi(I^2\mathbf{z})\mathbf{c}_1 + \Phi(I^2\mathbf{s})\mathbf{c}_2 + \Phi(I^2\mathbf{r})\mathbf{c}_3 - \mathbf{c}_3 + \Phi(I^2\mathbf{q})\mathbf{c}_4 + \Phi(I^2\mathbf{p})\mathbf{c}_5 \\ \Phi(I^3\mathbf{z})\mathbf{c}_1 + \Phi(I^3\mathbf{s})\mathbf{c}_2 + \Phi(I^3\mathbf{r})\mathbf{c}_3 + \Phi(I^3\mathbf{q})\mathbf{c}_4 - \mathbf{c}_4 + \Phi(I^3\mathbf{p})\mathbf{c}_5 \\ \Phi(I^4\mathbf{z})\mathbf{c}_1 + \Phi(I^4\mathbf{s})\mathbf{c}_2 + \Phi(I^4\mathbf{r})\mathbf{c}_3 + \Phi(I^4\mathbf{q})\mathbf{c}_4 + \Phi(I^4\mathbf{p})\mathbf{c}_5 - \mathbf{c}_5 \end{bmatrix} \\ &= \begin{bmatrix} \Phi(\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5) - \mathbf{c}_1 \\ \Phi\left(I(\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5)\right) - \mathbf{c}_2 \\ \Phi\left(I^2(\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5)\right) - \mathbf{c}_3 \\ \Phi\left(I^3(\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5)\right) - \mathbf{c}_4 \\ \Phi\left(I^4(\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5)\right) - \mathbf{c}_5 \end{bmatrix} = - \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{c}_4 \\ \mathbf{c}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{9}$$

Then $\mathbf{c}_i = \mathbf{0}$, $i = 1, ..., 5$ and so $\mathbf{c} = \mathbf{0}$. But by hypothesis $\mathbf{c} \neq \mathbf{0}$. So $u_0 \neq \mathbf{0}$. That $u_0 \in \ker B_4$ is proved as in [7]. Thus if B_4 is an injective operator then $\det \mathbf{V} \neq 0$. Statments (ii), (iii) are proved as in Theorem 2 [7].

Below we generalize Theorem 3.4 [9], where in the case of a Hilbert space H was proved the correctness and selfadjointness of the operator B_3 corresponding to the boundary value problem:

$$B_3x = \widehat{A}^3x - Y\langle \widehat{A}x, F^t\rangle_{H^m} - S\langle \widehat{A}^2x, F^t\rangle_{H^m} - G\langle \widehat{A}^3x, F^t\rangle_{H^m} = f,$$
$$D(B_3) = D(\widehat{A}^3),$$

where $\widehat{A}$ is a correct selfadjoint operator, $Y = \widehat{A}^2G - S\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} - G\overline{\langle F^t, \widehat{A}^2G\rangle}_{H^m}$, $S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}$ and C is a Hermitian $m \times m$ matrix.

Theorem 1.3 Let the space X, the operator $\widehat{A}$ and its inverse operator $I = \widehat{A}^{-1}$, and the vectors Φ, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ as above. Let the operator $B_3 : X \to X$ be defined by

$$\begin{aligned} B_3u &= \widehat{A}^3u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2u) - \mathbf{s}\Phi(\widehat{A}^3u) = f, \\ D(B_3) &= D(\widehat{A}^3), \end{aligned} \tag{10}$$

where $f \in X$. Then the following statements are true:
(i) The operator B_3 is injective if and only if

$$\det \mathbf{L} \neq 0,$$

where

$$\mathbf{L} = \begin{bmatrix} \Phi(\mathbf{s}) - \mathbf{1}_m & \Phi(\mathbf{r}) & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{s}) & \Phi(I\mathbf{r}) - \mathbf{1}_m & \Phi(I\mathbf{q}) & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) & \Phi(I^2\mathbf{q}) - \mathbf{1}_m & \Phi(I^2\mathbf{p}) \\ \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{q}) & \Phi(I^3\mathbf{p}) - \mathbf{1}_m \end{bmatrix}. \tag{11}$$

(ii) If the operator B_3 is injective, then B_3 is bijective and the unique solution to (10) for any $f \in X$ is given by

$$u = B_3^{-1}f = I^3f - \begin{pmatrix} I^3\mathbf{s} & I^3\mathbf{r} & I^3\mathbf{q} & I^3\mathbf{p} \end{pmatrix} \mathbf{L}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2f) \\ \Phi(I^3f) \end{pmatrix}. \tag{12}$$

(iii) If the inverse operator $I = \widehat{A}^{-1}$ is bounded on X, that is, $\widehat{A}$ is correct, then the operator B_3 is correct.
Proof. (i) Let $\det \mathbf{L} \neq 0$ and $u \in \ker B_3$. Then

$$\widehat{A}^3u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2u) - \mathbf{s}\Phi(\widehat{A}^3u) = 0, \quad u \in D(\widehat{A}^3). \tag{13}$$

By applying the inverse operator $I = \widehat{A}^{-1}$ on the both sides of (13) and on the equations following from (13), we get

$$\begin{aligned} &\widehat{A}^2u - I\mathbf{p}\Phi(u) - I\mathbf{q}\Phi(\widehat{A}u) - I\mathbf{r}\Phi(\widehat{A}^2u) - I\mathbf{s}\Phi(\widehat{A}^3u) = 0, \\ &\widehat{A}u - I^2\mathbf{p}\Phi(u) - I^2\mathbf{q}\Phi(\widehat{A}u) - I^2\mathbf{r}\Phi(\widehat{A}^2u) - I^2\mathbf{s}\Phi(\widehat{A}^3u) = 0, \\ &u - I^3\mathbf{p}\Phi(u) - I^3\mathbf{q}\Phi(\widehat{A}u) - I^3\mathbf{r}\Phi(\widehat{A}^2u) - I^3\mathbf{s}\Phi(\widehat{A}^3u) = 0. \end{aligned} \tag{14}$$

Now applying the functional Φ on the both sides of (13) and the above system, we obtain the system

$$\begin{aligned} &[\Phi(\mathbf{s}) - \mathbf{1}_m]\Phi(\widehat{A}^3u) + \Phi(\mathbf{r})\Phi(\widehat{A}^2u) + \Phi(\mathbf{q})\Phi(\widehat{A}u) + \Phi(\mathbf{p})\Phi(u) = 0, \\ &\Phi(I\mathbf{s})\Phi(\widehat{A}^3u) + [\Phi(I\mathbf{r}) - \mathbf{1}_m]\Phi(\widehat{A}^2u) + \Phi(I\mathbf{q})\Phi(\widehat{A}u) + \Phi(I\mathbf{p})\Phi(u) = 0, \\ &\Phi(I^2\mathbf{s})\Phi(\widehat{A}^3u) + \Phi(I^2\mathbf{r})\Phi(\widehat{A}^2u) + [\Phi(I^2\mathbf{q}) - \mathbf{1}_m]\Phi(\widehat{A}u) + \Phi(I^2\mathbf{p})\Phi(u) = 0, \\ &\Phi(I^3\mathbf{s})\Phi(\widehat{A}^3u) + \Phi(I^3\mathbf{r})\Phi(\widehat{A}^2u) + \Phi(I^3\mathbf{q})\Phi(\widehat{A}u) + [\Phi(I^3\mathbf{p}) - \mathbf{1}_m]\Phi(u) = 0, \end{aligned}$$

or

$$\mathbf{L} \begin{pmatrix} \Phi(\widehat{A}^3u) \\ \Phi(\widehat{A}^2u) \\ \Phi(\widehat{A}u) \\ \Phi(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{15}$$

where the matrix $\mathbf{L}$ is given in (11). Then, since $\det \mathbf{L} \neq 0$, we get

$$\Phi(\widehat{A}^3u) = \Phi(\widehat{A}^2u) = \Phi(\widehat{A}u) = \Phi(u) = 0. \tag{16}$$

Substitution (16) into (14) implies that $u = 0$. Thus $\ker B_3 = \{0\}$ and hence B_3 is an injective operator. Conversely. We prove that if B_3 is an injective operator then $\det \mathbf{L} \neq 0$, or equivalently, if $\det \mathbf{L} = 0$, then B_3 is not injective. Let $\det \mathbf{L} = 0$. Then there exists a nonzero vector of constants $\mathbf{c} = col(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4)$, where $\mathbf{c}_i = col(c_{i1}, ..., c_{im})$, $i = 1, ..., 4$ such that $\mathbf{Lc} = \mathbf{0}$. Consider the element $u_0 = I^3(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4) \in D(\widehat{A}^4)$. Note that $u_0 \neq 0$, because alternatively $\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4 = 0$ and then from $\mathbf{Lc} = \mathbf{0}$ follows that $\mathbf{Lc} =$

$$= \begin{bmatrix} \Phi(\mathbf{s}) - \mathbf{1}_m & \Phi(\mathbf{r}) & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{s}) & \Phi(I\mathbf{r}) - \mathbf{1}_m & \Phi(I\mathbf{q}) & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) & \Phi(I^2\mathbf{q}) - \mathbf{1}_m & \Phi(I^2\mathbf{p}) \\ \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{q}) & \Phi(I^3\mathbf{p}) - \mathbf{1}_m \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{c}_4 \end{bmatrix}$$

$$= \begin{bmatrix} \Phi(\mathbf{s})\mathbf{c}_1 - \mathbf{c}_1 + \Phi(\mathbf{r})\mathbf{c}_2 + \Phi(\mathbf{q})\mathbf{c}_3 + \Phi(\mathbf{p})\mathbf{c}_4 \\ \Phi(I\mathbf{s})\mathbf{c}_1 + \Phi(I\mathbf{r})\mathbf{c}_2 - \mathbf{c}_2 + \Phi(I\mathbf{q})\mathbf{c}_3 + \Phi(I\mathbf{p})\mathbf{c}_4 \\ \Phi(I^2\mathbf{s})\mathbf{c}_1 + \Phi(I^2\mathbf{r})\mathbf{c}_2 + \Phi(I^2\mathbf{q})\mathbf{c}_3 - \mathbf{c}_3 + \Phi(I^2\mathbf{p})\mathbf{c}_4 \\ \Phi(I^3\mathbf{s})\mathbf{c}_1 + \Phi(I^3\mathbf{r})\mathbf{c}_2 + \Phi(I^3\mathbf{q})\mathbf{c}_3 + \Phi(I^3\mathbf{p})\mathbf{c}_4 - \mathbf{c}_4 \end{bmatrix}$$

$$= \begin{bmatrix} \Phi(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4) - \mathbf{c}_1 \\ \Phi\left(I(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4)\right) - \mathbf{c}_2 \\ \Phi\left(I^2(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4)\right) - \mathbf{c}_3 \\ \Phi\left(I^3(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4\right) - \mathbf{c}_4 \end{bmatrix} = - \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{c}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Then $\mathbf{c}_i = \mathbf{0}$, $i = 1, ..., 4$ and thus we obtain $\mathbf{c} = \mathbf{0}$. Remind that by hypothesis $\mathbf{c} \neq \mathbf{0}$. So $u_0 \neq 0$. We will prove that $u_0 \in \ker B_3$. Indeed

$$\begin{gathered} B_3 u_0 = \widehat{A}^3 u_0 - \mathbf{p}\Phi(u_0) - \mathbf{q}\Phi(\widehat{A}u_0) - \mathbf{r}\Phi(\widehat{A}^2 u_0) - \mathbf{s}\Phi(\widehat{A}^3 u_0) \\ = \mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4 - \mathbf{p}\Phi\left(I^3(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4)\right) \\ -\mathbf{q}\Phi\left(I^2(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4)\right) - \mathbf{r}\Phi\left(I(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4)\right) \\ -\mathbf{s}\Phi\left(\mathbf{s}\mathbf{c}_1 + \mathbf{r}\mathbf{c}_2 + \mathbf{q}\mathbf{c}_3 + \mathbf{p}\mathbf{c}_4\right) \\ = -(\mathbf{s}, \mathbf{r}, \mathbf{q}, \mathbf{p})\mathbf{L}col(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4) = -(\mathbf{s}, \mathbf{r}, \mathbf{q}, \mathbf{p})\mathbf{Lc} = -(\mathbf{s}, \mathbf{r}, \mathbf{q}, \mathbf{p})\mathbf{0} = 0. \end{gathered}$$

Then $\ker B_3 \neq \{0\}$ and B_3 is not injective. Thus we proved that B_3 is an injective operator if and only if $\det \mathbf{L} \neq 0$.
(ii) From (i) follows that $\det \mathbf{L} \neq 0$. Then problem 10 has a unique solution. Acting as in the proof of (i) for any $f \in X$ we get

$$\widehat{A}^2 u - I\mathbf{p}\Phi(u) - I\mathbf{q}\Phi(\widehat{A}u) - I\mathbf{r}\Phi(\widehat{A}^2 u) - I\mathbf{s}\Phi(\widehat{A}^3 u) = If,$$

$$\widehat{A}u - I^2\mathbf{p}\Phi(u) - I^2\mathbf{q}\Phi(\widehat{A}u) - I^2\mathbf{r}\Phi(\widehat{A}^2 u) - I^2\mathbf{s}\Phi(\widehat{A}^3 u) = I^2 f, \tag{17}$$

$$u - I^3\mathbf{p}\Phi(u) - I^3\mathbf{q}\Phi(\widehat{A}u) - I^3\mathbf{r}\Phi(\widehat{A}^2 u) - I^3\mathbf{s}\Phi(\widehat{A}^3 u) = I^3 f. \tag{18}$$

Now applying the functional Φ on the both sides of (13) and (18), we obtain

$$\begin{gathered} [\Phi(\mathbf{s}) - \mathbf{1}_m]\Phi(\widehat{A}^3 u) + \Phi(\mathbf{r})\Phi(\widehat{A}^2 u) + \Phi(\mathbf{q})\Phi(\widehat{A}u) + \Phi(\mathbf{p})\Phi(u) = -\Phi(f), \\ -\Phi(I\mathbf{s})\Phi(\widehat{A}^3 u) + [\Phi(I\mathbf{r}) - \mathbf{1}_m]\Phi(\widehat{A}^2 u) + \Phi(I\mathbf{q})\Phi(\widehat{A}u) + \Phi(I\mathbf{p})\Phi(u) = -\Phi(If), \\ -\Phi(I^2\mathbf{s})\Phi(\widehat{A}^3 u) + \Phi(I^2\mathbf{r})\Phi(\widehat{A}^2 u) + [\Phi(I^2\mathbf{q}) - \mathbf{1}_m]\Phi(\widehat{A}u) + \Phi(I^2\mathbf{p})\Phi(u) = -\Phi(I^2 f), \\ -\Phi(I^3\mathbf{s})\Phi(\widehat{A}^3 u) + \Phi(I^3\mathbf{r})\Phi(\widehat{A}^2 u) + \Phi(I^3\mathbf{q})\Phi(\widehat{A}u) + [\Phi(I^3\mathbf{p}) - \mathbf{1}_m]\Phi(u) = -\Phi(I^3 f), \end{gathered}$$

or

$$\mathbf{V} \begin{pmatrix} \Phi(\widehat{A}^3 u) \\ \Phi(\widehat{A}^2 u) \\ \Phi(\widehat{A}u) \\ \Phi(u) \end{pmatrix} = - \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2 f) \\ \Phi(I^3 f) \end{pmatrix}.$$

The last equation gives

$$\begin{pmatrix} \Phi(\widehat{A}^3 u) \\ \Phi(\widehat{A}^2 u) \\ \Phi(\widehat{A}u) \\ \Phi(u) \end{pmatrix} = -\mathbf{L}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2 f) \\ \Phi(I^3 f) \end{pmatrix}. \tag{19}$$

Substituting (19) into (18), we get the unique solution (12) to problem (10). Since this solution holds for any $f \in X$, then $R(B_3) = X$, which means that B_3 is bijective.
(iii) If the inverse operator $I = \widehat{A}^{-1}$ is bounded, then the operator B_3^{-1}, defined by (12) is bounded, since the operator $\widehat{A}^{-1}$ and the components of the vector Φ are bounded. Hence, B_3 is correct.

Example 1.1 The operator $B_1 : C[0,1] \to C[0,1]$ which corresponds to the problem

$$\begin{aligned} u'''(t) &- 30t\int_0^1 xu(x)dx - 10(t^2+1)\int_0^1 xu'(x)dx - t^3\int_0^1 xu''(x)dx \\ &= t^3+7t^2+9t-5, \\ u(0) &= u'(0)=u''(0)=0, \end{aligned} \tag{20}$$

is correct and the unique solution to (20), for every $f \in C[0,1]$, is given by

$$u = t^4 - 2t^3. \tag{21}$$

Proof We refer to Theorem 1.3. If we compare equation (20) with equation (10) it is natural to take $\widehat{A}^3u = u'''(t)$, with $D(\widehat{A}^3) = \{u(t) :\in C^3[0,1] : u(0) = u'(0) = u''(0) = 0\}$, and so $\widehat{A}$ is defined by $\widehat{A}u = u'(t)$, $D(\widehat{A}) = \{u(t) \in C^1[0,1] : u(0) = 0\}$. Then $\widehat{A}^2u = u''(t)$, $D(\widehat{A}^2) = \{u(t) \in C^2[0,1] : u(0) = u'(0) = 0\}$. It is easy to verify that for any $f \in C[0,1]$

$$\widehat{A}^{-1}f(t) = If = \int_0^t f(x)dx, \quad \widehat{A}^{-2}f = I^2f = I(If), \quad \widehat{A}^{-3}f = I^3f = I(I^2f).$$

Also we can take $\mathbf{p} = p = 30t$, $\mathbf{q} = 10(t^2+1)$, $\mathbf{r} = r = t^3$, $\mathbf{s} = s = 0$,
$\Phi(u) = \int_0^1 xu(x)dx$, $\Phi(\widehat{A}u) = \int_0^1 xu'(x)dx$, $\Phi(\widehat{A}^2u) = \int_0^1 xu''(x)dx$, $f = t^3+7t^2+31t-5$. Then by Derive programm, we compute $Is = I^2s = I^3s = \Phi(s) = \Phi(Is) = \Phi(I^2s) = \Phi(I^3s) = 0$,
$Ir = \int_0^t x^3dx = \frac{t^4}{4}$, $Iq = \frac{10t^3}{3} + 10t$, $Ip = 15t^2$,
$I^2r = I(Ir) = \int_0^t x^3dx = \frac{t^5}{20}$, $I^2q = I(Iq) = \frac{5t^4}{6} + 5t^2$, $I^2p = I(Ip) = 5t^3$,
$I^3r = I(I^2r) = \frac{t^6}{120}$, $I^3q = I(I^2q) = \frac{t^5}{6} + \frac{5}{3}t^3$, $I^3p = I(I^2p) = \frac{5}{4}t^4$,
$\Phi(r) = \int_0^1 x(x^3)dx = 1/5$, $\Phi(q) = 10\int_0^1 x(x^2+1)dx = 15/2$,
$\Phi(p) = 30\int_0^1 x^2dx = 10$,
$\Phi(Ir) = \frac{1}{4}\int_0^1 x(x^4)dx = \frac{1}{24}$, $\Phi(Iq) = \int_0^1 x(\frac{10x^3}{3} + 10x)dx = 4$,
$\Phi(Ip) = 15\int_0^1 x^3dx = 15/4$,
$\Phi(I^2r) = \frac{1}{140}$, $\Phi(I^2q) = 25/18$, $\Phi(I^2p) = 1$,
$\Phi(I^3r) = \frac{1}{960}$, $\Phi(I^3q) = 5/14$, $\Phi(I^3p) = 5/24$. Since

$$\det\mathbf{L} = \begin{vmatrix} \Phi(\mathbf{s}) - \mathbf{1}_m & \Phi(\mathbf{r}) & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{s}) & \Phi(I\mathbf{r}) - \mathbf{1}_m & \Phi(I\mathbf{q}) & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) & \Phi(I^2\mathbf{q}) - \mathbf{1}_m & \Phi(I^2\mathbf{p}) \\ \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{q}) & \Phi(I^3\mathbf{p}) - \mathbf{1}_m \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1/5 & 15/2 & 10 \\ 0 & 1/24-1 & 4 & 15/4 \\ 0 & 1/140 & 25/18-1 & 1 \\ 0 & 1/960 & 5/14 & 5/24-1 \end{vmatrix} \neq 0,$$

problem (20), by Theorem 1.3, is correct. Further we compute
$If = \int_0^t (x^3+7x^2+31x-5)dx = \frac{1}{4}t^4 + \frac{7}{3}t^3 + \frac{31}{2}t^2 - 5t$,
$I^2f = I(If) = \frac{1}{20}t^5 + \frac{7}{12}t^4 + \frac{31}{6}t^3 - \frac{5}{2}t^2$,
$I^3f = I(I^2f) = \frac{1}{120}t^6 + \frac{7}{60}t^5 + \frac{31}{24}t^4 - \frac{5}{6}t^3$,
$\Phi(f) = 587/60$, $\Phi(If) = 163/60$, $\Phi(I^2f) = 323/630$, $\Phi(I^3f) = 191/2880$. Substituting the abow values into (12), we obtain (21).

References

[1] Apreutesei N., Ducrot A., Volpert V. Travelling waves for integro-differential equations in population dynamics. *Discrete and Continuous Dynamical Systems*, 2009, Ser. B, vol. 11, no. 3, pp. 541–561. DOI: 10.3934/dcdsb.2009.11.541 [in English].

[2] Baiburin M.M., Providas E. Exact Solution to Systems of Linear First-Order Integro-Differential Equations with Multipoint and Integral Conditions. In: Rassias T., Pardalos P. (eds) *Mathematical Analysis and Applications. Springer Optimization and Its Applications book series, volume 154*, 2019, pp. 1-16. DOI: https://doi.org/10.1007/978-3-030-31339-5_1 [in English].

[3] Bloom F. Ill-Posed Problems for Integrodifferential Equations in Mechanics and Electromagnetic Theory. *SIAM Studies in Applied Mathematics, Philadelphia*, 1981, 231 P. ISBN: 0-89871-171-1 Available at: http://bookre.org/reader?file=725637 [in English].

[4] Cushing J.M. Integrodifferential equations and delay models in population dynamics. *Springer-Verlag Berlin Heidelberg*, 1977. DOI: https://doi.org/10.1002/bimj.4710210608 [in English].

[5] Medlock J., Kot M. Spreading disease: integro-differential equations old and new. *Mathematical Biosciences*, August 2003, vol. 184, pp. 201–222. DOI: https://doi.org/10.1016/S0025-5564(03)00041-5 [in English].

[6] Oinarov R.O., Parasidi I.N. Correct extensions of operators with finite defect in Banach spases. *Izvestiya Akademii Nauk Kazakhskoi SSR*, 1988, vol. 5, pp. 42-46 [in Russian].

[7] Parasidis I.N., Providas E. Integro-differential equations embodying powers of a differential operator. *Vestnik Samarskogo universiteta. Estestvennonauchnaya seriya* [Vestnik of Samara University. Natural Science Series], 2019, vol. 25, no. 3, pp. 13–21. DOI: https://doi.org/10.18287/2541-7525-2019-25-3-12-21 [in English].

[8] Parasidis I.N., Providas E. On the Exact Solution of Nonlinear Integro-Differential Equations. In: *Applications of Nonlinear Analysis*, 2018, pp. 591–609. DOI: 10.1007/978-3-319-89815-5_21 [in English].

[9] ParasidisI.N., Tsekrekos P.C., Lokkas Th.G. Correct and self-adjoint problems for biquadratic operators. *Journal of Mathematical Sciences*, 2010, vol. 166, issue 2, pp. 420–427. DOI: https://doi.org/10.1007/s10958-011-0621-2 [in English].

[10] Parasidis I.N., Providas E. Extension Operator Method for the Exact Solution of Integro-Differential Equations. In: Pardalos P., Rassias T. (eds) Contributions in Mathematics and Engineering. Springer, Cham, 2016, pp. 473–496. DOI: https://doi.org/10.1007/978-3-319-31317-7_23 [in English].

[11] Polyanin A.D., Zhurov A.I. Exact solutions to some classes of nonlinear integral, integro-functional, and integro-differential equations. *Doklady Mathematics*, 2008, issue 77, pp. 315–319. DOI: https://doi.org/10.1134/S1064562408020403 [in English].

[12] Sachs E.W., Strauss A.K. Efficient solution of a partial integro-differential equation in finance. *Applied Numerical Mathematics*, 2008, issue 58, pp. 1687–1703. DOI: https://doi.org/10.1016/j.apnum.2007.11.002 [in English].

[13] Shishkin G.A. Linear Fredholm integro-differential equations. *Ulan-Ude, Buryat State University*, 2007. [in Russian].

[14] Shivanian E. Analysis of meshless local radial point interpolation (MLRPI) on a nonlinear partial integro-differential equation arising in population dynamics. *Engineering Analysis with Boundary Elements*, 2003, vol. 37, pp. 1693–1702. DOI: https://doi.org/10.1016/j.enganabound.2013.10.002 [in English].

[15] Vassiliev N.N., Parasidis I.N., Providas E. Exact solution method for Fredholm integro-differential equations with multipoint and integral boundary conditions. Part 1. Extention method. *Information and Control Systems*, 2018, issue 6, pp. 14–23. DOI: https://doi.org/10.31799/1684-8853-2018-6-14-23 [in English].

[16] Vassiliev N.N., Parasidis I.N., Providas E. Exact solution method for Fredholm integro-differential equations with multipoint and integral boundary conditions. Part 2. Decomposition-extension method for squared operators. *Information and Control Systems*, 2019, issue 2, pp. 2–9. DOI: https://doi.org/10.31799/1684-8853-2019-2-2-9 [in English].

[17] Wazwaz A.M. Linear and nonlinear integral equations, methods and applications. Berlin, Heidelberg: Springer, 2011. DOI: https://doi.org/10.1007/978-3-642-21449-3 [in English].