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СТЕПЕНИ НЕКОТОРОГО ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА**

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**АННОТАЦИЯ**

Установлены критерии разрешимости и корректности для двух линейных интегро-дифференциальных операторов типа Фредгольма  $B_2, B_4$ , включающих вплоть до второй и четвертой степени, соответственно дифференциального оператора  $\hat{A}$  с неизвестным обратным  $I = \hat{A}^{-1}$ . Представлены явные формулы решения соответствующих начальных и краевых задач при использовании обратного дифференциального оператора. Подход основан на теории продолжения линейных операторов в банаховых пространствах. В качестве примера решены три задачи для обыкновенных и частных интегро-дифференциальных операторов.

**Ключевые слова:** интегро-дифференциальные уравнения, задачи Коши, краевые задачи, дифференциальные операторы, степенные операторы, составные произведения, точные решения.

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## INTEGRO-DIFFERENTIAL EQUATIONS EMBODYING POWERS OF A DIFFERENTIAL OPERATOR

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### ABSTRACT

We establish solvability and correctness criteria for two Fredholm type linear integro-differential operators  $B_2, B_4$  encompassing up to second and fourth powers, respectively, of a differential operator  $\hat{A}$  with a known inverse  $I = \hat{A}^{-1}$ . We also derive explicit solution formulae to corresponding initial and boundary value problems by using the inverse of the differential operator. The approach is based on the theory of the extensions of linear operators in Banach spaces. Three example problems for ordinary and partial integro-differential operators are solved.

**Key words:** integro-differential equations, initial value problems, boundary value problems, differential operators, power operators, composite products, exact solutions.

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## Introduction

Let  $\mathbf{C}$  denote the set of all complex numbers and  $X, Y$  be complex Banach spaces. Let  $P : X \rightarrow Y$  be a linear operator and  $D(P)$  and  $R(P)$  its domain and range, respectively. We recall that a linear operator  $P : X \rightarrow Y$  is said to be *injective (or uniquely solvable)* if for all  $u_1, u_2 \in D(P)$  such that  $Pu_1 = Pu_2$ , follows that  $u_1 = u_2$ ; alternatively, the operator  $P$  is injective if and only if  $\ker P = \{0\}$ . A linear operator  $P : X \rightarrow Y$  is called *surjective (or everywhere solvable)* if  $R(P) = Y$ . The operator  $P$  is called *bijective* if  $P$  is both injective and surjective. Lastly,  $P$  is said to be *correct* if  $P$  is bijective and its inverse  $P^{-1}$  is bounded on  $Y$ .

Let  $X = Y$  and let the bijective operator  $\hat{A} : X \rightarrow X$ . We consider the power operators  $\hat{A}^2 = \hat{A}\hat{A}$  and  $\hat{A}^4 = \hat{A}^2\hat{A}^2$  defined as composite products, and the perturbed linear operators  $B_2 : X \rightarrow X$ ,  $B_4 : X \rightarrow X$  defined by

$$B_2u = \hat{A}^2u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\hat{A}u) - \mathbf{r}\Phi(\hat{A}^2u), \quad (1)$$

$$B_4u = \hat{A}^4u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\hat{A}u) - \mathbf{r}\Phi(\hat{A}^2u) - \mathbf{s}\Phi(\hat{A}^3u) - \mathbf{z}\Phi(\hat{A}^4u), \quad (2)$$

with  $D(B_2) = D(\hat{A}^2)$  and  $D(B_4) = D(\hat{A}^4)$ , respectively. The column vector

$$\Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix}, \quad \Phi(u) = \begin{pmatrix} \Phi_1(u) \\ \vdots \\ \Phi_m(u) \end{pmatrix}, \quad (3)$$

is a set of complex-valued, linear and bounded functionals  $\Phi_j : X \rightarrow \mathbf{C}$ ,  $j = 1, \dots, m$ , i.e.  $\Phi_j \in X^*$  and  $\Phi \in X_m^*$ , where  $X^*$  is the adjoint space of  $X$ . The row vectors

$$\mathbf{p} = (p_1 \ \cdots \ p_m), \quad \mathbf{q} = (q_1 \ \cdots \ q_m), \quad \mathbf{r} = (r_1 \ \cdots \ r_m), \quad (4)$$

$$\mathbf{s} = (s_1 \ \cdots \ s_m), \quad \mathbf{z} = (z_1 \ \cdots \ z_m), \quad (5)$$

are sets of elements  $p_j, q_j, r_j, s_j, z_j \in X$ ,  $j = 1, \dots, m$ , i.e.  $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z} \in X_m$ .

For later usage, we mention here that

$$\Phi(\mathbf{p}) = \begin{bmatrix} \Phi_1(p_1) & \cdots & \Phi_1(p_m) \\ \vdots & \ddots & \vdots \\ \Phi_m(p_1) & \cdots & \Phi_m(p_m) \end{bmatrix}, \quad (6)$$

is the  $m \times m$  matrix whose  $i, j$ -th entry  $\Phi_i(p_j)$  is the value of the functional  $\Phi_i$  on element  $p_j$ . Also, we note that  $\Phi(\mathbf{p}\mathbf{N}) = \Phi(\mathbf{p})\mathbf{N}$ , where  $\mathbf{N}$  is a  $m \times k$ ,  $k = 1, 2, \dots$ , constant matrix. Lastly,  $\mathbf{1}_m$  symbolizes the  $m \times m$  identity matrix and  $\mathbf{0}$  the zero column vector.

In the case where  $\hat{A}$  is a linear differential operator of order  $n$  and the functionals  $\Phi_j$ ,  $j = 1, \dots, m$ , designate Fredholm integral operators with separable kernels, then  $B_2, B_4$  describe Fredholm linear integro-differential operators. Integro-differential equations play an important role in modeling physical phenomena and processes in various disciplines in engineering, physics, biology, population dynamics, epidemiology, finance and others. Initial and boundary value problems for integro-differential equations are usually solved by numerical methods due to their complexity. Closed form solutions are obtained only for a limited number of problems, see for example in [5], [6], [9], [10] and the recent works by the authors [2], [3] [4], [7], [8].

In this paper, we are concerned with the solvability and the construction of the solution in closed form of the following two ordinary or partial integro-differential equations subject to initial or boundary conditions, which have not been studied before, namely

$$B_2 u = f, \quad D(B_2) = D(\hat{A}^2), \quad (7)$$

$$B_4 u = f, \quad D(B_4) = D(\hat{A}^4), \quad (8)$$

for any  $f \in X$ . Our approach is based on the theory of the extensions of linear operators in Banach spaces [1]. Problems (7), (8) are solved by using the inverse  $I = \hat{A}^{-1}$ .

The rest of the paper is organized as follows. In Section 1., the theory is developed and two main theorems are shown. In Section 2., the theory is applied to solve several example problems. Finally, some conclusions are stated in Section 2..

## 1. Main Results

We first derive solvability and correctness criteria for the operator  $B_2$  and construct the exact solution to initial and boundary value problems involving  $B_2$ . We state the following theorem.

**Theorem 1.** *Let  $X$  be a complex Banach space,  $\hat{A} : X \rightarrow X$  a bijective linear operator and  $I = \hat{A}^{-1}$  its inverse,  $\Phi \in X_m^*$ , and  $\mathbf{p}, \mathbf{q}, \mathbf{r} \in X_m$ . Let the operator  $B_2 : X \rightarrow X$  be defined by*

$$\begin{aligned} B_2 u &= \hat{A}^2 u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\hat{A}u) - \mathbf{r}\Phi(\hat{A}^2 u) = f, \\ D(B_2) &= D(\hat{A}^2), \end{aligned} \quad (9)$$

where  $f \in X$ . The following statements are true:

(i) If

$$\det \mathbf{W} = \det \begin{bmatrix} \Phi(\mathbf{r}) - \mathbf{1}_m & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) - \mathbf{1}_m & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{r}) & \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) - \mathbf{1}_m \end{bmatrix} \neq 0, \quad (10)$$

then the operator  $B_2$  is injective and everywhere solvable (bijective). The unique solution to (9) for any  $f \in X$  is given by

$$\begin{aligned} u &= B_2^{-1} f = \\ &= I^2 f - (I^2 \mathbf{r} \quad I^2 \mathbf{q} \quad I^2 \mathbf{p}) \mathbf{W}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(I f) \\ \Phi(I^2 f) \end{pmatrix}. \end{aligned} \quad (11)$$

(ii) If the operator  $B_2$  is injective and the vectors  $\mathbf{p}, \mathbf{q}, \mathbf{r}$  are linearly independent, then  $\det \mathbf{W} \neq 0$ .

(iii) If the inverse operator  $I = \hat{A}^{-1}$  is bounded on  $X$ , that is,  $\hat{A}$  is correct, then the operator  $B_2$  is correct.

*Proof.* (i) Let  $\det \mathbf{W} \neq 0$  and  $u \in \ker B_2$ , i.e.

$$\hat{A}^2 u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\hat{A}u) - \mathbf{r}\Phi(\hat{A}^2 u) = 0, \quad (12)$$

where  $u \in D(\widehat{A}^2)$ . By applying the inverse operator  $I = \widehat{A}^{-1}$  twice on both sides of (12), we obtain in succession

$$\widehat{A}u - I\mathbf{p}\Phi(u) - I\mathbf{q}\Phi(\widehat{A}u) - I\mathbf{r}\Phi(\widehat{A}^2u) = 0, \quad (13)$$

$$u - I^2\mathbf{p}\Phi(u) - I^2\mathbf{q}\Phi(\widehat{A}u) - I^2\mathbf{r}\Phi(\widehat{A}^2u) = 0. \quad (14)$$

Acting now by the functional vector  $\Phi$  on the both sides of (12)-(14), we obtain the following system of equations

$$\mathbf{W} \begin{pmatrix} \Phi(\widehat{A}^2u) \\ \Phi(\widehat{A}u) \\ \Phi(u) \end{pmatrix} = \mathbf{0}, \quad (15)$$

where the  $3m \times 3m$  matrix

$$\mathbf{W} = \begin{bmatrix} \Phi(\mathbf{r}) - \mathbf{1}_m & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) - \mathbf{1}_m & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{r}) & \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) - \mathbf{1}_m \end{bmatrix}. \quad (16)$$

Since  $\det \mathbf{W} \neq 0$ , it is concluded that

$$\Phi(\widehat{A}^2u) = \Phi(\widehat{A}u) = \Phi(u) = \mathbf{0}. \quad (17)$$

Substitution of (17) into (14) yields  $u = 0$ . Thus,  $\ker B_2 = \{0\}$  and therefore  $B_2$  is an injective operator.

To find the solution to problem (9), we work as above. By applying the inverse operator  $I = \widehat{A}^{-1}$  twice on both sides of (9), we get successively

$$\widehat{A}u - I\mathbf{p}\Phi(u) - I\mathbf{q}\Phi(\widehat{A}u) - I\mathbf{r}\Phi(\widehat{A}^2u) = If, \quad (18)$$

$$u - I^2\mathbf{p}\Phi(u) - I^2\mathbf{q}\Phi(\widehat{A}u) - I^2\mathbf{r}\Phi(\widehat{A}^2u) = I^2f. \quad (19)$$

Then acting by the vector of functionals  $\Phi$  on both sides of (9), (18), (19), we acquire the system

$$\mathbf{W} \begin{pmatrix} \Phi(\widehat{A}^2u) \\ \Phi(\widehat{A}u) \\ \Phi(u) \end{pmatrix} = - \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2f) \end{pmatrix}. \quad (20)$$

By inverting (20), we obtain

$$\begin{pmatrix} \Phi(\widehat{A}^2u) \\ \Phi(\widehat{A}u) \\ \Phi(u) \end{pmatrix} = -\mathbf{W}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2f) \end{pmatrix}. \quad (21)$$

Putting (19) into the form

$$u = I^2f + \begin{pmatrix} I^2\mathbf{r} & I^2\mathbf{q} & I^2\mathbf{p} \end{pmatrix} \begin{pmatrix} \Phi(\widehat{A}^2u) \\ \Phi(\widehat{A}u) \\ \Phi(u) \end{pmatrix}, \quad (22)$$

and then substituting (21) into (22), we obtain formula (11) which is the unique solution of the problem (9).

Finally, because  $f$  in (11) is an arbitrary element of  $X$ , it is implied that  $R(B_2) = X$ . Hence  $B_2$  is surjective.

(ii) We prove that if  $B_2$  is an injective operator then  $\det \mathbf{W} \neq 0$ , or equivalently, if  $\det \mathbf{W} = 0$  then  $B_2$  is not injective. Let  $\det \mathbf{W} = 0$ . Then there exists a nonzero vector of constants  $\mathbf{c} = \text{col}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ , where  $\mathbf{c}_i = \text{col}(c_{i1}, \dots, c_{im})$ ,  $i = 1, 2, 3$ , such that

$$\mathbf{W}\mathbf{c} = \mathbf{0}. \quad (23)$$

Consider the element  $u_0 = I^2(\mathbf{r}\mathbf{c}_1 + \mathbf{q}\mathbf{c}_2 + \mathbf{p}\mathbf{c}_3) \in D(\widehat{A}^2)$ . It follows that

$$\begin{aligned} B_2u_0 &= \widehat{A}^2u_0 - \mathbf{p}\Phi(u_0) - \mathbf{q}\Phi(\widehat{A}u_0) - \mathbf{r}\Phi(\widehat{A}^2u_0) = \\ &= - \begin{pmatrix} \mathbf{r} & \mathbf{q} & \mathbf{p} \end{pmatrix} \mathbf{W}\mathbf{c} = 0, \end{aligned} \quad (24)$$

by taking into account (23). This means that  $u_0 \in \ker B_2$ . Note that  $u_0 \neq 0$ , because by hypothesis  $\mathbf{p}, \mathbf{q}, \mathbf{r}$  are linearly independent and  $\mathbf{c} \neq \mathbf{0}$ . Therefore  $B_2$  is not injective.

(iii) In (11), the functionals of the vector  $\Phi$  are bounded. From the hypothesis that  $\widehat{A}$  is correct, it is implied that  $I = \widehat{A}^{-1}$  and  $I^2$  are bounded on  $X$ . Therefore the operator  $B_2^{-1}$  is bounded on  $X$ , and from (i) follows that  $B_2$  is correct. The theorem is proved.  $\square$

*Remark 1.* In the cases where one or two of the vectors  $\mathbf{p}, \mathbf{q}, \mathbf{r}$  are equal to zero vector, then analogous results to Theorem 1 can be derived. In practice, we can obtain the solution formula directly from (11)

by removing the corresponding columns and rows. For instance, let us assume that  $\mathbf{r} = \mathbf{0}$ . In this case the problem (9) degenerates to the following one,

$$\begin{aligned} B_2 u &= \widehat{A}^2 u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) = f, \\ D(B_2) &= D(\widehat{A}^2). \end{aligned} \quad (25)$$

Then the operator  $B_2$  is injective if

$$\det \mathbf{W} = \det \begin{bmatrix} \Phi(I\mathbf{q}) - \mathbf{1}_m & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) - \mathbf{1}_m \end{bmatrix} \neq 0, \quad (26)$$

and the unique solution of (25) for any  $f \in X$  is given by

$$\begin{aligned} u &= B_2^{-1} f = \\ &= I^2 f - (I^2\mathbf{q} \ I^2\mathbf{p}) \mathbf{W}^{-1} \begin{pmatrix} \Phi(I f) \\ \Phi(I^2 f) \end{pmatrix}. \end{aligned} \quad (27)$$

Next, we elaborate on the solvability and correctness of the operator  $B_4$  and the exact solution of initial and boundary value problems incorporating  $B_4$ . We show the subsequent theorem.

**Theorem 2.** *Let  $X$  be a complex Banach space,  $\widehat{A} : X \rightarrow X$  a bijective operator and  $I = \widehat{A}^{-1}$  its inverse,  $\Phi \in X_m^*$ , and  $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z} \in X_m$ . Let the operator  $B_4 : X \rightarrow X$  be defined by*

$$\begin{aligned} B_4 u &= \widehat{A}^4 u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) = f, \\ D(B_4) &= D(\widehat{A}^4), \end{aligned} \quad (28)$$

where  $f \in X$ . Then the following statements are true:

(i) If

$$\begin{aligned} \det \mathbf{V} &= \det \begin{bmatrix} \Phi(\mathbf{z}) - \mathbf{1}_m & \Phi(\mathbf{s}) & \Phi(\mathbf{r}) \\ \Phi(I\mathbf{z}) & \Phi(I\mathbf{s}) - \mathbf{1}_m & \Phi(I\mathbf{r}) \\ \Phi(I^2\mathbf{z}) & \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) - \mathbf{1}_m \\ \Phi(I^3\mathbf{z}) & \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) \\ \Phi(I^4\mathbf{z}) & \Phi(I^4\mathbf{s}) & \Phi(I^4\mathbf{r}) \\ & \Phi(\mathbf{q}) & \Phi(\mathbf{p}) \\ & \Phi(I\mathbf{q}) & \Phi(I\mathbf{p}) \\ & \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) \\ & \Phi(I^3\mathbf{q}) - \mathbf{1}_m & \Phi(I^3\mathbf{p}) \\ & \Phi(I^4\mathbf{q}) & \Phi(I^4\mathbf{p}) - \mathbf{1}_m \end{bmatrix} \\ &\neq 0, \end{aligned} \quad (29)$$

then the operator  $B_4$  is injective and everywhere solvable on  $X$  (bijective). The unique solution to the problem (28) for any  $f \in X$  is given by

$$\begin{aligned} u &= B_4^{-1} f = \\ &= I^4 f - (I^4\mathbf{z} \ I^4\mathbf{s} \ I^4\mathbf{r} \ I^4\mathbf{q} \ I^4\mathbf{p}) \mathbf{V}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(I f) \\ \Phi(I^2 f) \\ \Phi(I^3 f) \\ \Phi(I^4 f) \end{pmatrix}. \end{aligned} \quad (30)$$

(ii) If the operator  $B_4$  is injective and the vectors  $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z}$  are linearly independent, then  $\det \mathbf{W} \neq 0$ .

(iii) If the inverse  $I = \widehat{A}^{-1}$  is bounded on  $X$ , i.e.  $\widehat{A}$  is correct, then the operator  $B_4$  is correct.

*Proof.* (i) Let  $\det \mathbf{V} \neq 0$  and  $u \in \ker B_4$ , i.e.

$$\widehat{A}^4 u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) = 0, \quad (31)$$

where  $u \in D(\widehat{A}^4)$ . By applying the inverse operator  $I = \widehat{A}^{-1}$  four times on both sides of (31), we get consecutively

$$\widehat{A}^3 u - I \left( \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) \right) = 0, \quad (32)$$

$$\widehat{A}^2 u - I^2 \left( \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) \right) = 0, \quad (33)$$

$$\widehat{A} u - I^3 \left( \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) \right) = 0, \quad (34)$$

$$u - I^4 \left( \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) \right) = 0. \quad (35)$$

Implementing the vector of functionals  $\Phi$  on both sides of (31)–(35), we obtain the system of equations

$$\mathbf{V} \begin{pmatrix} \Phi(\widehat{A}^4 u) \\ \Phi(\widehat{A}^3 u) \\ \Phi(\widehat{A}^2 u) \\ \Phi(\widehat{A} u) \\ \Phi(u) \end{pmatrix} = \mathbf{0}, \quad (36)$$

where the  $5m \times 5m$  matrix  $\mathbf{V}$  is given in (29). Then, since  $\det \mathbf{V} \neq 0$ , we acquire

$$\Phi(\widehat{A}^4 u) = \Phi(\widehat{A}^3 u) = \Phi(\widehat{A}^2 u) = \Phi(\widehat{A} u) = \Phi(u) = \mathbf{0}. \quad (37)$$

Substitution of (37) into (35) yields  $u = 0$ . Thus,  $\ker B_4 = \{0\}$  and hence  $B_4$  is an injective operator.

To obtain the solution of (28) we work in similar manner. By applying the inverse operator  $I = \widehat{A}^{-1}$  on both sides of (28) four successive times, we get

$$\widehat{A}^3 u - I(\mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u)) = If, \quad (38)$$

$$\widehat{A}^2 u - I^2(\mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u)) = I^2 f, \quad (39)$$

$$\widehat{A} u - I^3(\mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u)) = I^3 f, \quad (40)$$

$$u - I^4(\mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u)) = I^4 f. \quad (41)$$

Acting by the functional vector  $\Phi$  on both sides of (28) and (38)–(41), we obtain the system

$$\mathbf{V} \begin{pmatrix} \Phi(\widehat{A}^4 u) \\ \Phi(\widehat{A}^3 u) \\ \Phi(\widehat{A}^2 u) \\ \Phi(\widehat{A} u) \\ \Phi(u) \end{pmatrix} = - \begin{pmatrix} \Phi(f) \\ \Phi(If) \\ \Phi(I^2 f) \\ \Phi(I^3 f) \\ \Phi(I^4 f) \end{pmatrix}. \quad (42)$$

We write (41) in matrix form

$$u = I^4 f + \begin{pmatrix} I^4 \mathbf{z} & I^4 \mathbf{s} & I^4 \mathbf{r} & I^4 \mathbf{q} & I^4 \mathbf{p} \end{pmatrix} \begin{pmatrix} \Phi(\widehat{A}^4 u) \\ \Phi(\widehat{A}^3 u) \\ \Phi(\widehat{A}^2 u) \\ \Phi(\widehat{A} u) \\ \Phi(u) \end{pmatrix}. \quad (43)$$

By inverting (42) and substituting into (43), we get the unique solution (30) of the problem (28).

Lastly,  $f$  in (30) is an arbitrary element of  $X$  and therefore  $R(B_4) = X$ . Hence the operator  $B_4$  is surjective.

(ii) We prove that if  $B_4$  is an injective operator then  $\det \mathbf{V} \neq 0$ , or equivalently, if  $\det \mathbf{V} = 0$  then  $B_4$  is not injective. Let  $\det \mathbf{V} = 0$ . Then there exists a nonzero vector  $\mathbf{c} = \text{col}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5)$ , where  $\mathbf{c}_i = \text{col}(c_{i1}, \dots, c_{im})$ ,  $i = 1, \dots, 5$ , such that

$$\mathbf{V}\mathbf{c} = \mathbf{0}. \quad (44)$$

Consider the element  $u_0 = I^4(\mathbf{z}\mathbf{c}_1 + \mathbf{s}\mathbf{c}_2 + \mathbf{r}\mathbf{c}_3 + \mathbf{q}\mathbf{c}_4 + \mathbf{p}\mathbf{c}_5) \in D(\widehat{A}^4)$ . Then, we have

$$\begin{aligned} B_4 u_0 &= \widehat{A}^4 u_0 - \mathbf{p}\Phi(u_0) - \mathbf{q}\Phi(\widehat{A}u_0) - \mathbf{r}\Phi(\widehat{A}^2 u_0) - \mathbf{s}\Phi(\widehat{A}^3 u_0) - \mathbf{z}\Phi(\widehat{A}^4 u_0) = \\ &= -(\mathbf{z} \ \mathbf{s} \ \mathbf{r} \ \mathbf{q} \ \mathbf{p}) \mathbf{V}\mathbf{c} = 0, \end{aligned} \quad (45)$$

by making use of (44). This means that  $u_0 \in \ker B_4$ . Note that  $u_0 \neq 0$  since by hypothesis the vectors  $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z}$  are linearly independent and  $\mathbf{c} \neq \mathbf{0}$ . Hence  $B_4$  is not injective.

(iii) Since the functionals in vector  $\Phi$  and the operators  $I, I^i$ ,  $i = 2, 3, 4$ , are bounded on  $X$ , it is concluded that the operator  $B_4^{-1}$  is bounded too and from (i) follows that  $B_2$  is correct. The theorem is proved.  $\square$

*Remark 2.* In the cases where one or more of the vectors  $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z}$  are zero, then similar results to Theorem 2 can be obtained. Actually, we can have the corresponding solution formula directly from (30) by removing the like columns and rows. For example, suppose that  $\mathbf{p} = \mathbf{0}$ . Then the problem (28) reduces to

$$\begin{aligned} B_4 u &= \widehat{A}^4 u - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u) = f, \\ D(B_4) &= D(\widehat{A}^4). \end{aligned} \quad (46)$$

Then the operator  $B_4$  is injective if  $\det \mathbf{V} \neq 0$ , where

$$\mathbf{V} = \begin{bmatrix} \Phi(\mathbf{z}) - \mathbf{1}_m & \Phi(\mathbf{s}) & \Phi(\mathbf{r}) & \Phi(\mathbf{q}) \\ \Phi(I\mathbf{z}) & \Phi(I\mathbf{s}) - \mathbf{1}_m & \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) \\ \Phi(I^2\mathbf{z}) & \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) - \mathbf{1}_m & \Phi(I^2\mathbf{q}) \\ \Phi(I^3\mathbf{z}) & \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{q}) - \mathbf{1}_m \end{bmatrix}. \quad (47)$$

The unique solution of the problem (46) for every  $f \in X$  is given by

$$\begin{aligned} u &= B_4^{-1}f = \\ &= I^4 f - (I^4\mathbf{z} \ I^4\mathbf{s} \ I^4\mathbf{r} \ I^4\mathbf{q}) \mathbf{V}^{-1} \begin{pmatrix} \Phi(f) \\ \Phi(I f) \\ \Phi(I^2 f) \\ \Phi(I^3 f) \end{pmatrix}. \end{aligned} \quad (48)$$

## 2. Applications

**Example 1.** Consider the following problem

$$\begin{aligned} z''(x) - \lambda \int_{-1}^1 x[z'(t) + z(t)]dt &= x, \quad x \in [-1, 1], \\ z(0) = 1, \quad z'(0) &= 0. \end{aligned} \quad (49)$$

Making the substitution  $u(x) = z(x) - 1$ , we get

$$\begin{aligned} u''(x) - \lambda x \int_{-1}^1 [u'(t) + u(t)]dt &= (1 + 2\lambda)x, \quad x \in [-1, 1], \\ u(0) = u'(0) &= 0. \end{aligned} \quad (50)$$

We take  $X = C[-1, 1]$ , accordingly  $X^1 = C^1[-1, 1]$ ,  $X^2 = C^2[-1, 1]$ , and

$$\begin{aligned} \widehat{A}u &= u', \quad D(\widehat{A}) = \{u : u \in X^1, u(0) = 0\}, \\ \widehat{A}^2u &= u'', \quad D(\widehat{A}^2) = \{u : u \in X^2, u(0) = u'(0) = 0\}, \\ \Phi(u) &= (\Phi_1(u)) = \left( \int_{-1}^1 u(t)dt \right), \\ \Phi(\widehat{A}u) &= (\Phi_1(\widehat{A}u)) = \left( \int_{-1}^1 u'(t)dt \right), \\ \mathbf{p} = \mathbf{q} &= (\lambda x), \quad \mathbf{r} = \mathbf{0}, \\ f &= (1 + 2\lambda)x, \end{aligned} \quad (51)$$

$m = 1$ , and the operator  $B_2 : X \rightarrow X$  as

$$\begin{aligned} B_2u &= \widehat{A}^2u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u) = \\ &= u'' - \lambda x \int_{-1}^1 u(t)dt - \lambda x \int_{-1}^1 u'(t)dy = (1 + 2\lambda)x, \\ D(B_2) &= D(\widehat{A}^2) = \{u : u \in X^2, u(0) = u'(0) = 0\}. \end{aligned}$$

Note that  $\widehat{A}$  is injective and  $R(\widehat{A}) = X$ ,  $\Phi_1 \in X^*$ , and  $\mathbf{p}, \mathbf{q}$  are linearly dependent. It is known that the inverse operator  $I$  and its composite  $I^2$  are given by

$$I \cdot = \widehat{A}^{-1} \cdot = \int_0^x \cdot dt, \quad I^2 \cdot = \widehat{A}^{-2} \cdot = \int_0^x (x-t)dt, \quad (52)$$

and they are bounded on  $X$ . As stated by Remark 1 and by using the results in (51) and (52), we compute

$$\begin{aligned} \det \mathbf{W} &= \det \begin{bmatrix} \Phi(I\mathbf{q}) - 1 & \Phi(I\mathbf{p}) \\ \Phi(I^2\mathbf{q}) & \Phi(I^2\mathbf{p}) - 1 \end{bmatrix} = \\ &= \det \begin{bmatrix} \frac{\lambda}{3} - 1 & \frac{\lambda}{3} \\ 0 & -1 \end{bmatrix} = 1 - \frac{\lambda}{3}. \end{aligned}$$

From Theorem 1, it is implied that the operator  $B_2$  is correct if  $\lambda \neq 3$ . In this case the unique solution to the problem (50) is

$$u(x) = \frac{2\lambda + 1}{2(\lambda - 3)}x^3, \quad (53)$$

by means of (27), while the solution to the problem (49) follows from  $u(x) = z(x) - 1$ .

**Example 2.** Let  $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ ,  $\partial\Omega = \{x \in \mathbb{R}^3 : |x| = 1\}$  and  $\mathbb{H}^4(\Omega)$  the Sobolev space of all functions of  $L_2(\Omega)$  which have their partial generalized derivatives up to the fourth order Lebesgue integrable. Consider the problem,

$$\begin{aligned} \Delta^2 u(x) - g(x) \int_{\Omega} v(y) \Delta u(y) dy &= f(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} &= 0, \end{aligned} \quad (54)$$

where  $g(x), v(x), f(x) \in L_2(\Omega)$  are given functions and  $u(x) \in \mathbb{H}^4(\Omega)$  is the unknown function.

Comparing (54) with (9) in Theorem 1, we take  $X = L_2(\Omega)$  and

$$\begin{aligned} \widehat{A}u &= \Delta u, \quad D(\widehat{A}) = \{u : u \in \mathbb{H}^2(\Omega), u|_{\partial\Omega} = 0\}, \\ \widehat{A}^2 u &= \Delta^2 u, \quad D(\widehat{A}^2) = \{u : u \in \mathbb{H}^4(\Omega), u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0\}, \\ \Phi(\widehat{A}u) &= \left( \Phi_1(\widehat{A}u) \right) = \left( \int_{\Omega} v(x) \Delta u(x) dx \right), \\ \mathbf{p} = \mathbf{r} &\equiv \mathbf{0}, \quad \mathbf{q} = (g(x)), \end{aligned} \quad (55)$$

$m = 1$ , and the operator  $B_2 : X \rightarrow X$  defined as

$$B_2 u = \widehat{A}^2 u - \mathbf{q} \Phi(\widehat{A}u), \quad D(B_2) = D(\widehat{A}^2). \quad (56)$$

Since  $v(x) \in X$ , it is concluded that the functional  $\Phi_1$  is bounded on  $X$ , i.e.  $\Phi_1 \in X^*$ . The Dirichlet problem for Poisson equation

$$\Delta u(x) = f(x), \quad u(x)|_{\partial\Omega} = 0, \quad u \in \mathbb{H}^2(\Omega), \quad f \in X, \quad (57)$$

is known to be everywhere solvable and admits a unique solution for almost all  $x \in \Omega$ , viz.

$$u(x) = \widehat{A}^{-1} f(x) = \int_{\Omega} \mathcal{G}(x, y) f(y) dy, \quad \forall f \in X, \quad (58)$$

where  $\mathcal{G}(x, y)$  is Green's function. Thus, the operator  $\widehat{A}$  is bijective and

$$\begin{aligned} I \cdot &= \widehat{A}^{-1} \cdot = \int_{\Omega} \mathcal{G}(x, y) \cdot dy, \\ I^2 \cdot &= \widehat{A}^{-2} \cdot = \int_{\Omega} \mathcal{G}(x, y) \int_{\Omega} \mathcal{G}(y, t) \cdot dt dy. \end{aligned} \quad (59)$$

From Theorem 1, Remark 1 and the use of (55) and (59) we have

$$\det \mathbf{W} = \det [\Phi(I\mathbf{q}) - 1] = \int_{\Omega} v(x) \int_{\Omega} \mathcal{G}(x, y) g(y) dy dx - 1. \quad (60)$$

If  $\int_{\Omega} v(x) \int_{\Omega} \mathcal{G}(x, y) g(y) dy dx \neq 1$ , then the operator  $B_2$  is injective and the unique solution of the problem (54), for any  $f \in X$ , is obtained by substitution into

$$u = B_2^{-1} f = I^2 f - I^2 \mathbf{q} \mathbf{W}^{-1} \Phi(I f). \quad (61)$$

**Example 3.** Let the problem

$$\begin{aligned} u^{(4)}(x) - 48x^2 \int_0^1 (1-t)u'(t)dt - 15x^3 \int_0^1 (1-t)u''(t)dt - \\ - 8x \int_0^1 (1-t)u'''(t)dt - 3x^4 \int_0^1 (1-t)u^{(4)}(t)dt = \\ = x^4 + \frac{1}{2}x^3 + \frac{1}{3}x^2 + 2x - 1, \\ u(0) = u'(0) = u''(0) = u'''(0) = 0. \end{aligned} \quad (62)$$

We take  $X = C[0, 1]$ , suitably  $X^1 = C^1[0, 1]$ ,  $X^2 = C^2[0, 1]$ ,  $X^3 = C^3[0, 1]$ ,  $X^4 = C^4[0, 1]$ , and

$$\begin{aligned} \widehat{A}u &= u', \quad D(\widehat{A}) = \{u : u \in X^1, u(0) = 0\}, \\ \widehat{A}^2 u &= u'', \quad D(\widehat{A}^2) = \{u : u \in X^2, u(0) = u'(0) = 0\}, \\ \widehat{A}^3 u &= u''', \quad D(\widehat{A}^3) = \{u : u \in X^3, u(0) = u'(0) = u''(0) = 0\}, \\ \widehat{A}^4 u &= u^{(4)}, \quad D(\widehat{A}^4) = \{u : u \in X^4, u(0) = u'(0) = u''(0) = u'''(0) = 0\}, \\ \Phi(\widehat{A}u) &= \left( \Phi_1(\widehat{A}u) \right) = \left( \int_0^1 (1-t)u'(t)dt \right), \end{aligned}$$



$$\begin{aligned}
\Phi(\widehat{A}^2 u) &= \left( \Phi_1(\widehat{A}^2 u) \right) = \left( \int_0^1 (1-t)u''(t)dt \right), \\
\Phi(\widehat{A}^3 u) &= \left( \Phi_1(\widehat{A}^3 u) \right) = \left( \int_0^1 (1-t)u'''(t)dt \right), \\
\Phi(\widehat{A}^4 u) &= \left( \Phi_1(\widehat{A}^4 u) \right) = \left( \int_0^1 (1-t)u^{(4)}(t)dt \right), \\
\mathbf{p} &= \mathbf{0}, \quad \mathbf{q} = (48x^2), \quad \mathbf{r} = (15x^3), \quad \mathbf{s} = (8x), \quad \mathbf{z} = (3x^4), \\
f &= x^4 + \frac{1}{2}x^3 + \frac{1}{3}x^2 + 2x - 1,
\end{aligned} \tag{63}$$

$m = 1$ , and the operator  $B_4 : X \rightarrow X$  defined by

$$\begin{aligned}
B_4 u &= \widehat{A}^4 u - \mathbf{q}\Phi(\widehat{A}u) - \mathbf{r}\Phi(\widehat{A}^2 u) - \mathbf{s}\Phi(\widehat{A}^3 u) - \mathbf{z}\Phi(\widehat{A}^4 u), \\
D(B_4) &= D(\widehat{A}^4).
\end{aligned}$$

Notice that  $\widehat{A}$  is injective and  $R(\widehat{A}) = X$ ,  $\Phi_1 \in X^*$ , and  $\mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{z}$  are linearly independent. It is known from elementary books of differential equations that

$$I^k \cdot = \widehat{A}^{-k} \cdot = \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} \cdot dt, \quad k = 1, 2, 3, 4, \tag{64}$$

and they are bounded on  $X$ . According to Theorem 2 and specifically to Remark 2, we form the matrix  $\mathbf{V}$  and compute its determinant, viz.

$$\begin{aligned}
\det \mathbf{V} &= \det \begin{bmatrix} \Phi(\mathbf{z}) - 1 & \Phi(\mathbf{s}) & \Phi(\mathbf{r}) & \Phi(\mathbf{q}) \\ \Phi(I\mathbf{z}) & \Phi(I\mathbf{s}) - 1 & \Phi(I\mathbf{r}) & \Phi(I\mathbf{q}) \\ \Phi(I^2\mathbf{z}) & \Phi(I^2\mathbf{s}) & \Phi(I^2\mathbf{r}) - 1 & \Phi(I^2\mathbf{q}) \\ \Phi(I^3\mathbf{z}) & \Phi(I^3\mathbf{s}) & \Phi(I^3\mathbf{r}) & \Phi(I^3\mathbf{q}) - 1 \end{bmatrix} = \\
&= \det \begin{bmatrix} -\frac{9}{10} & \frac{4}{3} & \frac{3}{4} & 4 \\ \frac{1}{70} & -\frac{2}{3} & \frac{1}{8} & \frac{4}{5} \\ \frac{1}{560} & \frac{1}{15} & -\frac{55}{56} & \frac{2}{15} \\ \frac{1}{5040} & \frac{1}{90} & \frac{1}{448} & -\frac{103}{105} \end{bmatrix} = \\
&= \frac{240388859}{444528000} \neq 0.
\end{aligned}$$

Hence, from Theorem 2, Remark 2 and the use of (63) and (64) follows that the operator  $B_4$  is correct and the unique solution to the problem (62) is given analytically by

$$u(x) = \frac{(x-5)x^4}{120}. \tag{65}$$

## Conclusions

We have presented a method for constructing the exact solution to initial and boundary value problems for a class of integro-differential operators embodying powers of a correct differential operator. We have included several problems to demonstrate the applicability and efficiency of the method. The proposed solution technique can be easily incorporated to any computer algebra system and therefore it may be a useful tool to researchers and students.

In closing, we state without elaborating that under certain conditions the two problems discussed in the present paper can become of the kind,

$$\begin{aligned}
B_2 u &= B^2 u = f, \quad D(B_2) = D(\widehat{A}^2), \\
B_4 u &= B^4 u = f, \quad D(B_4) = D(\widehat{A}^4),
\end{aligned}$$

where the operator  $B : X \rightarrow X$  is defined by

$$Bu = \widehat{A}u - \mathbf{p}\Phi(u) - \mathbf{q}\Phi(\widehat{A}u), \quad D(B) = D(\widehat{A}), \tag{66}$$

to facilitate further the solution process by employing decomposition techniques.

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