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SHAPE PROPERTIES OF THE SPACE OF PROBABILITY MEASURES AND ITS SUBSPACES

In this article we consider covariant functors acting in the categorie of compacts, preserving the shapes of infinite compacts, ANR-systems, moving compacts, shape equivalence, homotopy equivalence and $A(N)SR$ properties of compacts. As well as shape properties of a compact space X consisting of connectedness components 0 of this compact X under the action of covariant functors, are considered. And we study the shapes equality $ShX = ShY$ of infinite compacts for the space $P(X)$ of probability measures and its subspaces.

Key words: Covariant functors, $A(N)R$-compacts, ANR-systems, probability measures, moving compacts, retracts, measures of finite support, shape equivalence, homotopy equivalence.

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For a compact X by $P(X)$denote the space of probability measures. It is known that for an infinite compact X, this space $P(X)$ is homeomorphic to the Hilbert cube Q. For a natural number $n \in N$ by $P_n(X)$ denote the set of all probability measures with no more than n support, i.e. $P_n(X) = \{\mu \in P(X) : |\mathrm{supp}\mu| \leqslant n\}$. The compact $P_n(X)$ is a convex linear combination of Dirac measures in the form

$$\mu = m_1\delta_{x_1} + m_2\delta_{x_2} + ... + m_n\delta_{x_n}, \sum_{i=1}^{n} m_i = 1, m_i \geqslant 0, x_i \in X,$$

δ_{x_i}- the Dirac measure at a point x_i. By $\delta(X)$ denote the set of all Dirac measures. Recall that the space $P_f(X) \subset P(X)$ consists of all probability measures in the form $\mu = m_1\delta_{x_1} + m_2\delta_{x_2} + ... + m_k\delta_{x_k}$ of finite support, for each of which $m_i \geqslant \frac{k}{k+1}$ for some i. For a positive integer n put $P_{f,n} \equiv P_f \cap P_n$. For a compact X we have $P_{f,n}(X) = \{\mu \in P_f(X) : |\mathrm{supp}\mu| \leqslant n\}$; $P_f^C \equiv P_f \cap P^C, P_{f,n}^C \equiv P_f \cap P_n \cap P^C.P_n^C \equiv P^C \cap P$. For the compact X by $P^C(X)$ denote the set of all measures $\mu \in P(X)$ the support of each of which lies in one of the components of the compact X [12].

1. Introduction

For a space X by $\square X$ denote the expansion (partition) of the space X consisting of all the connected components. If $f : X \to Y$ is a continuous mapping, then the continuous mapping $\square f : \square X \to \square Y$ is uniquely determined by condition $\pi_Y{}^\circ f = \square f \cdot \pi_X$, where $\pi_Y : Y \to \square Y$ and $\pi_X : X \to \square X$, i.e. we have the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & \square f & \downarrow \pi_Y \\ \square X & \to & \square Y \end{array} \tag{1.1}$$

Lemma 1. If X is a compact ANR-space, then the map $P^C(\pi_X)$ isahomotopy equivalence.

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Proof. Let be an ANR-compact, then the space $P^C(X)$ is a finite set or is a finite union of Hilbert cubes and points. The space $P^C(\Box X)$ consists of finitely many points, because the space X is an ANR-compact. For any $\mu \in P^C(\Box X)$ the transformation $(P^C(f))^{-1}(\mu)$ is the Hilbert cube, or one point, i.e. $Sh((P^C(f))^{-1}(\mu))$ is trivial, then by Theorem 7 [5] the map $P^C(f)$ is a shape equivalence, and thus, is a homotopy equivalence. The proof is complete.

Theorem 1. Let X be a compact and let $\pi_X : X \to \Box X$ be a quotient map. Then the mapping $P^C(\pi_X)$ induces a shape equivalence, i.e. $Sh\left(P^C(X)\right) = Sh(\Box X)$.

Proof. Suppose X is compact, $\Box X$ is also compact, then by V.I.Ponamareva theorem [6] $\dim \Box X = 0$. Hence, $\dim P^C(X) = 0$ and $P^C(\Box X) = 0$. By Theorem 2 [5] the mapping $P^C(\pi_X)$ is a shape equivalence. This means that $Sh\left(P^C(X)\right) = ShP^C(\Box X)$ and $\left|\Box P^C(\Box X)\right| = |\Box X|$. This proves the theorem.

Definition [10]. A normal subfunctor F of the functor P_n is called locally convex if the set $F(\widetilde{n})$ is locally convex.

We say that a functor F_1 is a subfunctor (respectively nadfunktorom?) of a functor F_2 if there exists a natural transformation $h : F_1 \to F_2$ that the map $h(X) : F_1(X) \to F_2(X)$ is a monomorphism (epimorphism) for each object X. By exp denote the hyperspace functor of closed subsets. For example, the identity functor Id is a subfunctor of $\exp_n$, where $\exp_n X = \{F \in \exp X : |F| \leqslant n\}$, and the nth degree functor ^{n} is a nadfunktorom of functors $\exp_n$ and SP^n_G. A normal subfunctor F of the functor P_n is uniquely determined by its value $F(n)$ at an n-point space. Note that $P_n(n)$ is the $(n-1)$-dimensional simplex. Any subset of the $(n-1)$-dimensional simplex σ^{n-1} defines a normal subfunctor of the functor P_n if it is invariant under simplicial mappings.

An example of not normal subfunctor of the functor P_n is the functor of probability measures P^C_n whose supports lie in one of components. One of the examples of locally convex subfunctors of P_n, is a functor $SP^n \equiv SP^n_{S_n}$.

Corollary 1. If for compacts X and Y the equality $|\Box X| = |\Box Y| = \aleph_0$ holds, then $Sh\left(P^C(X)\right) = Sh\left(P^C(Y)\right)$ and $ShP(X) = ShP(Y)$, where $|Z|$ is the cardinality of a set Z.

Proof. Suppose the sets $|\Box X|$ and $|\Box Y|$ are countable. In this case, by Arkhangelskii's result [8], the spaces $|\Box X|$ and $|\Box Y|$ are compact and metrizable. Note that $|\Box X|$ and $|\Box Y|$ have a dense set of isolated points. Then the compacts $P(X)$ and $P(Y)$ are homeomorphic to the Hilbert cube Q. On the other hand, $P^C[X] = \Box X$ and $P^C[\Box Y] = \Box Y$. Consequently, $Sh\left(P^C(\Box X)\right) = Sh\left(P^C(\Box Y)\right)$. The corollary is proved.

By $M_\Box$ we denote the class of all compacts X such that $\Box X$ is metrizable. From corollary it follows that if $X, Y \in M_\Box$, then $\Box X$ and $\Box Y$ have a countable dense set of isolated points [9].

Corollary 2. If $X, Y \in M_\Box$, then either $Sh\left(P^C(X)\right) \geqslant Sh\left(P^C(Y)\right)$ or $Sh\left(P^C(X)\right) \leqslant Sh\left(P^C(Y)\right)$. Therefore, if $\Box X$ and $\Box Y$ are infinite, then $Sh\left(P^C(X)\right) = Sh\left(P^C(Y)\right)$, i.e. $Sh\left(P^C(X)\right) \geqslant Sh\left(P^C(Y)\right)$ and $Sh\left(P^C(X)\right) \leqslant Sh\left(P^C(Y)\right)$.

Proof. Suppose that X and Y are elements of the family $M_\Box$. Then $\Box X$ and $\Box Y$ are the zero-dimensional compacta. In particular, if $\Box X$ and $\Box Y$ are finite sets, then by Theorem 1 we obtain the desired.

If $|\Box X| \geqslant \aleph_0$, then $\Box X$ contains Cantor's discontinuum. In this case, $\Box Y$ can be embedded into $\Box X$, then the compact $\Box Y$ is a retract for $\Box X$ [10]. $Sh(\Box X) \geqslant Sh(\Box Y)$ and $Sh\left(P^C(\Box X)\right) \geqslant Sh\left(P^C(\Box Y)\right)$.

Consequently, by Theorem 1 we have $ShP^C[\Box X] \geqslant ShP^C[\Box Y]$. If $\Box X \leqslant \aleph_0$ and $\Box Y \leqslant \aleph_0$, then compacts $\Box X$ and $\Box Y$ are homeomorphic to Mazurkiewicz-Sierpinski ordinal compact [11]. Last, suppose $\Box X$ and $\Box Y$ are infinite sets, then $Sh(\Box X) \geqslant Sh(\Box Y)$ if and only if $\Box X$ and $\Box Y$ are homeomorphic [3]. If $|\Box X| > |\Box Y|$ or $|\Box X| < |\Box Y|$, then either $\Box Y$ or $\Box X$ is retract for $\Box X$or $\Box Y$, respectively. By Theorem 1 we have $Sh\left(P^C(X)\right) \geqslant Sh\left(P^C(Y)\right)$. Corollary 2 is proved.

Remark. In [11] it is shown that the Borsuk's definition of shapes of compacts is equivalent to the shapes of ANR-systems.

Lemma 2. For any compact X we have $|\Box P_f(X)| = |\Box X|$.

Proof. Let X b an arbitrary compact, $\Box X$ its set of connected components, i.e. $\Box X = \{x'_i \in X : \pi_X^{-1}(x'_i) -$ is connected component of the point $x'_i\}$. It is obvious that $\Box X$ is compact and $\Box X \subset X$. Hence, $Sh(\Box X) \leqslant ShX$. On the other hand, the commutativeness of the diagram

$$\begin{array}{c} \pi_X : X \to \Box X \\ \uparrow \qquad \uparrow \\ P_f(\pi_X) : P_f(X) \to \delta(\Box X) \end{array} \tag{1.2}$$

implies $|\Box ShP_f(X)| = |\Box X|$. From (1.2) we get $|\Box P_f(X)| = |\Box X|$. Lemma 2 is proved.

Let us note that for all $x \in X$ and $y \in X$ between sets $\left(r_f^{-1}\right)(x)$ and $\left(r_f^{-1}\right)(y)$ there is a one-one correspondence, i.e. to an arbitrary point $\mu_x \in \left(P_f^{-1}\right)(X)$ we assign $\mu_y \in \left(P_f^x\right)^{-1}$, where

$$\mu_x = m_0\delta_{x_0} + m_1\delta_{x_1} + ... + m_k\delta_{x_k}, \mu_y = m_0\delta_{y_0} + ... + m_k\delta_{x_k}.$$

In the case of the infinite compacts X and Y the spaces $P(X)$ and $P(Y)$are homeomorphic to the Hilbert cube Q. If A and B are Z-sets lying in the compacts $P(X)$ and $P(Y)$, then by Chapman's theorem [2], $ShA = ShB$ if and only if $P(X)\backslash A$ is homeomorphic to $P(Y)\backslash B$. In [10,12] it is shown that the subspaces $F(X)$ and $F(Y)$are Z-sets in the compacts $P(X)$ and $P(Y)$, where $F = P_f(X), P_{f,n}(X), P_{f,n}^C(X), P_f^C(X)$. Moreover, it was noted that this space X is a strong deformation retract for $F(X)$. So the following is valid.

Theorem 2. For infinite compacts X and Y the following conditions are equivalent:

1. $ShX = ShY$;
2. $P(X)\backslash P_f(X) \simeq P(Y)\backslash P_f(Y)$;
3. $P(X)\backslash \delta(X) \simeq P(Y)\backslash \delta(Y)$;
4. $P(X)\backslash F(X) \simeq P(Y)\backslash F(Y)$, where $F = P_{f,n}^C, P_f^C$.

Theorem 3. Suppose that X and Y are elements of $M_\Box$, $X \in M_\Box$and $Y \in M_\Box$. Then the following conditions are equivalent:

1. $Sh(\Box X) = Sh(\Box Y)$;
2. $P(X)\backslash P^C(X) \simeq P(Y)\backslash P^C(Y)$.

Theorem 4. Suppose that X and Y are elements of $M_\Box$. Then $Sh(\Box X) = Sh(\Box Y)$ if and only if $ShX = Sh(\Box X)$.

It is known that from the inequality $ShX \leqslant ShY$ it follows $Sh(\Box X) \leqslant Sh(\Box Y)$. In particular, the equality $ShX = ShY$ implies $Sh(\Box X) = Sh(\Box Y)$.

Now let $Sh(\Box X) = Sh(\Box Y)$. From the fact that the compacts $\Box X$ and $\Box Y$ are zero-dimensional and metrizable, and by Mardeschicha Segal theorem [3], $\Box X$ and $\Box Y$ are homeomorphic. If for any $y \in \Box X$ the set $\pi_y^{-1}(y)$ has the trivial shape, then by Theorem 7 [5] we have $ShY = Sh(\Box X)$; By virtue of the zero-dimensionality and equality $ShY = Sh(\Box X)$ it follows $Y \simeq \Box X \simeq \Box Y$.

Note that in this case $ShX = ShY$ and $X \simeq Y$, i.e. $ShX = Sh(\Box X)$ is equivalent to $ShX = ShY$.

Corollary 3. a) The space $P^C(X)$ is an ASR if and only if X is connected; b) $P^C(X)$ is an $ANSR$ if and only if X has finitely many connected components.

Theorem 5. For any infinite zero-dimensional compacts X and Y the followings are true:
a) If $ShX = ShY$, then $P_n(X) \simeq P_n(Y)$;
b) if $ShX = ShY$, then $P(X)\backslash P_n(X) \simeq P(Y)\backslash P_n(Y)$;
c) $ShP_n(X) = ShP_n(Y)$ if and only if $P(X)\backslash P_n(X) \simeq P(Y)\backslash P_n(Y)$;
d) $ShF(X) = ShF(Y)$ if and only if $P(X)\backslash F(X) \simeq P(Y)\backslash F(Y)$, where $F$are locally convex subfunctors of the functor P_n;
e) $ShX = ShY$ if and only if $P(X)\backslash \delta(X) \simeq P(Y)\backslash \delta(Y)$.

Theorem 6. For any infinite zero-dimensional compacts X and Y the following conditions are equivalent:

1. $ShX = ShY$;
2. $ShF(X) = ShF(Y)$, where $F = P_{f,n}, P_{f,n}^C, P_f, P_f^C$;
3. $X \simeq Y$;
4. $P(X)\backslash F(X) \simeq P(Y)\backslash F(Y)$;

Theorem 7. For any infinite compacts X and Y we have: a) if $ShX = ShY$, then $P(X)\backslash P_n(X) \simeq P(Y)\backslash P_n(Y)$for any $n \in N$;
b) if $ShX = ShY$, then $P(X)\backslash F(X) \simeq P(Y)\backslash F(Y)$, where F are locally convex subfunctors of the functors P_n.

Theorem 8. For any infinite compacts $X \in M_\Box$ and $Y \in M_\Box$ we have:
a) $ShX = ShY$ if and only if $P(X)\backslash P_n(X) \simeq P(Y)\backslash P_n(Y)$;
b) $ShX = ShY$ if and only if $P(X)\backslash F(X) \simeq P(Y)\backslash F(Y)$.

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СВОЙСТВА ФОРМЫ ВЕРОЯТНОСТНОГО ПРОСТРАНСТВА И ЕГО ПОДПРОСТРАНСТВ

В этой заметке мы рассмотрим ковариантные функторы, действующие в категории компактов, сохраняющие формы бесконечных компактов, ANR-систем, движущиеся компакты, эквивалентность формы, гомотопическую эквивалентность и $A(N)SR$ свойства компактов. Рассмотрены свойства формы компактного пространства X, состоящего из компонент связности 0 этого компактного X под действием ковариантных функторов. И мы изучаем равенство форм $ShX = ShY$ бесконечных компактов для пространства вероятностных мер $P(X)$ и его подпространств.

Ключевые слова: Ковариантный функтор, шейп компакта, компонента, связности и гомотопическая эквивалентность.

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