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EQUIVARIANT PROPERTIES OF THE SPACE $\mathbb{Z}(X)$ FOR A STRATIFIABLE SPACE X

ABSTRACT

In this paper, we prove the action of the compact group G defined by the stratified space X is continuous to the space $Z(X)$ being a stratified space containing the self-stratified space X as a closed subset. An equivariant analogue of some results of R. Cauty concerning $A(N)R(S)$ – spaces is proved. It is presented that the orbit space $Z(X)/G$ by the action of the group G is a S space.

Key words: equivariant maps; stratified space; group actions; orbit space; invariant set; homotopy density; dimension; absolute extensor; neighborhood extensor; covariant functor; probabilistic measures.

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Introduction

In the category of stratifiable spaces and continuous images, we include one construction belonging to the test space [1; 2] that defines the covariant functor in this category. This construction defines the functor that allows each stratified space X to be immersed in a closed manner into some other space $Z(X)$, which is the stratified space with "good" functorial, geometric and topological properties.

A stratifiable spaces can be defined of as topological space that is divided into smooth manifolds. Then a stratification in this context [3] is a structure associated with the emergence of closed sets, which is locally a decomposed space. This is what [4] refer to as a “germinal stratification". Each decomposed space causes a bundle of germs-stratifications, hence the concepts are consistent. Another notion of stratification can be found in [4], where the boundary conditions are slightly different. Stratifiable spaces always admit tangent bundle. They are relevant because they are really singular, while the usual vector bundle is not [5]. Families of examples and applications [6; 7] arise from smooth equivariant vector bundles. The next main motivation for developing stratifiable vector bundles is to use them for quantization purposes. In particular, in the Kostant–Suriot–Weyl quantization picture, three components of the initial data are required: a symplectic manifold, a complex linear bundle with a connection, and polarization, all of which satisfy various compatibility conditions [8]. Therefore, it is relevant to consider and define theorems for equivariant properties of spaces over a stratifiable space.

Let X be a stratified (briefly, $S-$space) space. For each open subset U of the space X and any point $x \in U$ of the set U we put:
a) $n(U,x) = \min\{m : x \in U_m\}$, where $U = \bigcup_{k=1}^{\infty} U_k$;
b) $U_x = U_{n(U,\ \ x)} \backslash (\overline{X\backslash\{x\}})_{n(U,x)}$.

Obviously, the set U_x is an open neighborhood of the point x and $U_x \subset U$. The set U_x has the following properties:

1°. U_x is an open neighborhood of the point x;

2°. If $U_x \cap V_y \neq \emptyset$ and $n(U,x) \leqslant n(V,y)$ then $y \in U$.

3°. If $U_x \cap V_y \neq \emptyset$ then $x \in V$ or $y \in U$.

Let X be the topological space, $|F(X)|$ be a complete simplicial complex whose vertices are points in the space X, i.e. $\left|F(X)^0\right| = X$. The space $|F(X)|$ has a weak topology. Now we define the topology on the space $|F(X)|$, the bases of open sets of which we denote by $Z(X)$ consists of W open in $F(X)$, satisfies following conditions:

o1. $W \cap X$ is an open in X;

o2. $|F(W \cap X)| \subset W$;

i. e. $\tau_{Z(X)} = \{W \in \tau_{|F(X)|} : W$ satisfies the conditions o1-o2$\}$.

Condition o2 means that every simplex $\sigma \in F(X)$ is contained in W if all vertices of this simplex σ lie entirely in $W \cap X$.

1. Main results

For the subset $A \subset X$, the set $F(A)$ is a subcomplex of the full complex $F(X)$ and $Z(A)$ is a subspace of the space $Z(X)$. Obviously, $Z(A)$ is closed in $Z(A)$ if A is closed in X.

For each $n \in \mathbb{Z}_+ = N \cup \{0\}$ we put $Z_n\,(X) = |F(X)^n|$ $Z_n(X)$ is a subspace of $Z\,(X)$. Then $Z_0(X) \cong X$ and $Z\,(X) = \bigcup_{n=0}^{\infty} Z_n\,(X)$. It is easy to see that for any $n \in Z_+$ the subspace $Z_n(X)$ is closed in $Z(X)$.

Let us introduce the following notation:

$T(A) = \{\sigma \in F(X)\backslash F(A) : \sigma \cap A \neq \emptyset\}$;

$\mathrm{M}\,(\mathrm{A}) = \{x \in \mathrm{Z}\,(\mathrm{X}) :$ exists $\sigma \in F\,(A)$ such that $x(\widehat{\sigma}) > 0\}$;

$T_n(A) = T(A) \cap (F(X)^n \backslash F(X)^{n-1})$;

$M_n(A) = Z(A) \cup (M(A) \cap Z_n(X)$;

For each $\varepsilon \in (0,1)^{T(A)}$ and for each $n \in N$, define the set:

$$M(A,\varepsilon) = \cup_{n\in\mathbb{Z}_+} M_n(A,\varepsilon),$$

where $M_0(A,\varepsilon) = Z(A) = |F(A)|$ and $M_n(A,\varepsilon) = Z(A) \cup \{\sigma(\varepsilon(\sigma)) \cap \pi_\sigma^{-1}(M_{n-1}(A,\varepsilon)) : \sigma \in T_n(A)\}$. Then the equality $M(A,\varepsilon) \cap X = A$ holds.

For each open set $\mathcal{U}$ of the space X, the set $M(\mathcal{U},\varepsilon)$ is open in $Z(X)$. In this case, the family $\mathscr{B}(M) =$ $= \{(\mathcal{U},\varepsilon) : \mathcal{U}$ is open in $\varepsilon\ \epsilon\ (0,1)^{(\mathcal{U})}\}$ is an open base of the space $Z(X)$.

Therefore, if for every $n \in N$ and every $\varepsilon \in (0,1)^{T_1(A)\cup T_2(A)\cup\ldots\cup T_n(A)}$ the set $M_n(A,\varepsilon)$ is defined, then the family

$\mathscr{B}(M) = \{M_1,(U,\varepsilon)\} : U$ is open in X and $\varepsilon \in (0,1)^{T_1(U)}\}$ is an open base for $Z_1(X)$, i.e. the following holds.

Lemma [9]. Families $\{M(U,\varepsilon) : U$ is open in X and $\varepsilon \in (0,1)^{T(U)}\}$ and $\{M_1(U,\varepsilon) : U$ is open in X and $\varepsilon \in (0,1)^{T_1(U)}\}$ is the base of the space $Z(X)$, (respectively, the space $Z_1(X)$).

In the work [2] R. Cauty claimed that for the space $Z(X)$ the following are true:

a) Each continuous map $f : A \to Y$, where A is a closed subset of the stratified space X, has a continuous extension to all X with values in $Z(Y)$: that is, the following diagram holds

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \curvearrowright & & \curvearrowright \\ X & \xrightarrow{\bar{f}} & Z(Y) \end{array}$$

$\tilde{f}|_A = f$. $X, Y \in S$.

b) The stratifiable space X is $AR(S)$ ($ANR(S)$) if and only if X is a retract (respectively, a neighborhood retract) of the space $Z(X)$.

Definition [6]. A topological space L is called hyper-connected (respectively, m -hyper-connected) if for each $i \in N$, there is a mapping $h_i : L^i \times \sigma^{i-1} \to L$ satisfying a, b and c (respectively, a, b and d):

a) $t \in \sigma^{n-1}$ and $t_i = 0$ implies $h_n(x,t) = h_{n-1}(\delta_i \cdot x, \delta_i \cdot t)$ for each $x \in L^n$ and $n = 2, 3, \ldots$

b) For each $x \in L^n$ the mapping $t \to L_n(x,t)$ maps the sets σ^{n-1} to L continuously;

c) For each $x \in L$ and a neighborhood U of x, there is a neighborhood V of x such that $\bigcup_{i=1}^{\infty} h_i(V^i \times \sigma^{i-1}) \subset U$ and $V \subset U$;

d) For each $x \in L$ and a neighborhood U of the point x, there is a neighborhood V of the point x such that $\bigcup_{i=1}^{n} h_i(V^i \times \sigma^{i-1}) \subset U$ and $V \subset U$, where $\sigma^{n-1} = \{t \in R^n : \sum_{i=1}^{n} t_i = 1, t_i \geqslant 0\}$ - $(n-1)$ is a dimensional simplex a $\delta_i : A^n \to A^{n-1}$ mapping defined by the formula $\delta_i(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ $i = \overline{1,n}$, i.e. δ_i – "forgetting" i-th coordinate of the product.

A space L is said to be a locally hyper-connected if for each point $x \in L$, there exists a neighborhood V of the point x such that V is hyper-connected.

In the paper [2] R. Cauty proved that $X \in A(N)R(S)$ if and only if X is hyperconnected (respectively, locally hyperconnected).

Theorem 1. For an arbitrary S-space X, the space $Z(X)\backslash X$ is a $AR(S)$ space.

Proof. Let $n \in N$. We construct mapping $h_n(z_1, \ldots, z_n, t) : (ZX\backslash X)^n \times \sigma^{n-1} \to ZX\backslash X$ assuming $h_n(z_1, z_2, \ldots, z_n)(t_1, t_2, \ldots, t_n) = \sum_{i=1}^{n} z_i t_i$, where $(z_1, z_2, \ldots, z_n) \in (ZX\backslash X)^n, (t_1, t_2, \ldots, t_n) \in \sigma^{n-1}$, $\sum_{i=1}^{n} t_i = 1, t_i \geqslant 0$.

It is easy to show that $h_n((z_1, \ldots, z_n) \times t) \in Z(X)\backslash X$.

Now we show that the space $Z(X)\backslash X$ is hyperconnected.

a) Let $t \in \sigma^{n-1}$, $t = (t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, t_{i+2}, \ldots, t_n)$. Then

$h_n(z,t) = h_n((z_1, z_2, \ldots, z_n)(t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, t_{i+2}, \ldots, t_n) = (t_1 z_1 + t_2 z_2 + \ldots + t_{i-1} z_{i-1} + t_{i+1} z_{i+1} + \ldots t_n z_n) = h_{n-1}((z_1, z_2, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n)(t_1, t_2, \ldots, t_n)) = h_{n-1}(\delta_i z, \delta_i t)$.

b) We fix $z_0 \in (Z(X)\backslash X)^n$, $z_0 = (z_1^0, z_2^0, \ldots, z_n^0)$, $z_1^0 \in ZX\backslash X$.

Hence $z_1^0 = \sum_{k=1}^{l_i} m_k^i, \overline{x_k^l}, \sum_{i=1}^{l_i} m_k^i = 1, m_k^i \geqslant 0$.

Let $t \in \sigma^{n-1}$, then $t \to h_n(z_0, t) = \sum_{i=1}^{n} t_i z_i^0 = t_1 \sum_{k=1}^{l_1} \mu_k \overline{x_k^1} + t_2 \sum_{k=1}^{l_2} \mu_k \overline{x_k^2} + \ldots + t_n \sum_{k=1}^{l_n} \mu_k \overline{x_k^n}$; Let us put $t_i \mu_j = a_{ij}$, $t_i \geqslant 0$, $\mu_j \geqslant 0$, $a_{ij} \geqslant 0$, $i = \overline{1,n}$, $j = l_1, \ldots, l_n$, $\sum_{ij} a_{ij} = 1$. Hence, we get $\sum_{i=1}^{n} \sum_{j=1}^{l_i} a_{ij} \overline{x_j^i}$. Consider the set $X_0 = \{x_j^i : i, j\}$.

In this case, point $h(z_0, t) \in Z(X_0)$, i.e. there is a simplex σ lying in $Z(X_0)$ whose vertices consist of points of the set X_0. On the other hand, if we consider the simplex σ^{n-1} with vertices $z_1^0, \ldots, z_n^0$, i.e. $\sigma_0^{n-1} = \langle z_1^0, z_2^0, \ldots, z_n^0 \rangle$. The mapping $h_n(z,t)$ with continuity in the argument t or the mapping $t \to h_n(z,t)$ completely covers the simplex σ_0^{n-1}, i.e. the mapping $t \to h_n(z,t)$ as a homeomorphism maps σ^{n-1} to z_0^{n-1}. Hence, the mapping $t \to h_n(z,t)$ is continuous.

c) Let $z_0 \in Z(X)\backslash X$ and U_{z_0} be an arbitrary neighborhood of the point z_0 in $Z(X)\backslash X$. Consider supp$z_0 = \{x_1, x_2, \ldots, x_k\}$ the support of the point z_0 of the space $Z(X)\backslash X$. Then $z_0 \in \langle x_1, x_2, \ldots, x_k \rangle = \sigma$. By the definition of topology in the space $Z(X)\backslash X$, the set $V^1 = \sigma \bigcap U_{z_0}$ is open. Consider a set V of the form $\{z \in \sigma \cap U_{z_0} = V^1 : \text{segment } [z, z_0] \subset V^1\}$. Obviously, the set V is open and convex. By definition, the following takes place: $V \subset V^1 \subset U_{z_0}$. Note that if $z \in (Z(X)\backslash X)^n$, $z = (z_1, z_2, \ldots, z_n)$ and supp$z_i \subset A, A \subset X$, then supp $h_n(z,t) \subseteq A$. If V is convex, then the maps $h_n(z,t)$ by definition maps $V^n \times \sigma^{n-1}$ to V. Therefore, the following holds: $\bigcup_{n=1}^{\infty} h_n(V^n \times \sigma^{n-1}) \subset U$. Hence, the space $Z(X)\backslash X$ is hyperslash. By virtue of R. Cauty's theorem [1], we obtain that $Z(X)\backslash X$ is $AR(S)$. Theorem 1 is proved.

Theorem 2. The finite product of $A(N)R(S)$ spaces is $A(N)R(S)$ spaces. Theorem 2 is proved in [6].

Let X be a topological space, G is a topological group $\theta : G \times X \to X$ is a continuous mapping such that

(1) $\theta(g, \theta(h,x)) = \theta(gh, x)$ for all $h \in G$ and $x \in X$;

(2) $\theta(e,x) = x$ for all $x \in X$, where e is the unit of the group G.

The mapping θ is called the action of the group G on the space X. The space X with a fixed action θ of the group G is called a G-space.

A set A is called invariant under the action of the group G (or G-invariant) if $G(A) = A$, where $G(A) = \{g(x) : g \in G, x \in A\}$.

For $g \in G$, we define the mapping $\theta_g : X \to X$ by the formula $\theta_g(x) = g(x) = \theta(g,x)$. By virtue of (1) or (2) we have $\theta_g \circ \theta_h = \theta_{gh}$ and θ_e is the identity mapping 1_X of the space X into itself. Thus, $\theta_g \circ \theta_{g-1} = \theta_e = 1_X = \theta_{g-1} \cdot \theta_g$ therefore, for $g \in G$, the mapping θ_g is a homeomorphism of the space X onto itself.

An action θ is called effective if $\text{Ker}\theta = e$ (i.e., the mapping θ is injective), where $\text{Ker}\theta = \{g \in G : g(x) = x\}$ for any $x \in X$ the kernel of the action of θ, and is almost effective if $\text{Ker}\theta$ – is a discrete subgroup of the group G. Obviously, the kernel $\text{Ker}\theta$ is a normal divisor of the group G and is closed in G.

We note, that some of the supporting statements were considered in the articles [10–14].

Definition [15]. Let X and Y be $G-$spaces.

The mapping $\varphi : X \to Y$ is called an equivariant mapping (or G-mapping) if φ commutes by actions, i.e. $\varphi(g(x)) = g(\varphi(x))$ for all $g \in G$ and all $x \in X$.

For a fixed group G, the class of G-spaces is a class of objects of a certain category, whose morphisms are called equivariant maps.

An equivariant mapping $\varphi : X \to Y$ that is a homeomorphism is called the G -equivalence of G -spaces.

Note that if we denote by $Homeo$(X) the group (with respect to, composition) of all homomorphisms of the space X onto itself. The mapping $g \to \theta_g$ defines a homeomorphism $\theta : G \to Homeo(X)$.

Let X be some G space, and let $x \in X$. The set $G_x = \{g \in G : g(x) = x\}$ of elements of the group, for which x is a fixed point, is obviously a closed subgroup of the group G. This subgroup G_x is called the stationary subgroup (or the stabilizer of the point x).

On the other hand, note that $\ker\theta$ is exactly $\bigcap_{x\in X} G_x$, i.e. $\ker\theta = \bigcap_{x\in X} G_x$.

The action of the group G on the space X is called free if for any point $x \in X$ the subgroup G_x is trivial. An action is called semi-free if the stationary subgroup G_x of any point $x \in X$ is either trivial or is the whole of G. Take $x \in X$. The subspace $G(x) = \{g(x) : g \in G\}$ is called the orbit of the point x (with respect to the action of the group G). Note that $G(x) \subset X$ for any $x \in X$ and for points x and y the sets $G(x)$ and $G(y)$ either do not intersect each other or coincide, i.e. $G(x) \bigcap G(y) = \emptyset$ or $G(x) = G(y)$ for any $x, y \in X$. By $X\backslash G = \{G(x) : x \in X\}$ we denote the orbit set of $G-$space X. Let $\pi = \pi_X : X \to X/G-$ be a natural mapping, associating the point x and the orbit $x^* = G(x)$. Then $X\backslash G$ is endowed with the quotient topology in the usual way (i.e., the set $U \subset X\backslash G$ is open if and only if $\pi^{-1}(U)$ is open in X), and the resulting topological space is called the orbit space. Note that if $U \subset X$ is open, then the set $G(U) = \bigcup_{g\in G} g(U)$ is open, since each of the sets $g(U) = \theta_g(U)$ (recall that $\theta_g : X \to X$ is a homeomorphism).

Therefore, for an open $U \subset X$, the set $\pi^{-1}\pi(U) = G(U)$ is also open, which by definition means that the set $\pi(U)$ is open in $X\backslash G$. Hence the projection $\pi : X \to X/G$ is a continuous open map.

Theorem [15]. Let the group G be compact and X is some $G-$ space. Then

(1) the space $X\backslash G$ is Hausdorff;

(2) The projection $\pi : X \to X/G$ is a closed map;

(3) The projection $\pi : X \to X/G$ is a proper mapping (that is, the preimage of any compact set is compact);

(4) The compactness of the space X is equivalent to the compactness of the space $X\backslash G$;

(5) The local compactness of the space X is equivalent to the local compactness of the space $X\backslash G$.

Let X be a stratified G -space that is the topological group G acts on the space X, i.e. there is a continuous mapping $(G, X) : G \times X \to X$ defined by the formula: $(g, x) = gx$. On the test space $Z(X)$, the action of the group G is defined as follows: $(G,\ Z(X)) : G \times Z(X) \to Z(X)$ $g \in G$, $z \in Z(X)$, $z = \sum_{i=1}^{k} m_i\overline{x_i}$, $\sum_{i=1}^{k} m_i = 1$, $m_i \geqslant 0$ $(g, z) = g \cdot z = \sum_{i=1}^{k} m_i\overline{g(x_i)}$. Thus, the space $Z(X)$ is a $G-$ space. It is easy to see that the space X in the G space $Z(X)$ is an invariant G-subset, i.e. if $x \in X$, then $g(x) = x \in X$.

Thus, the following holds.

Theorem 3. A continuous action of the group G defined on the space X extends continuously to the entire space $Z(X)$. Take the point $z = \sum_{i=1}^{k} m_i\overline{x_i}$, $m_i \geqslant 0$, $\sum_{i=1}^{k} m_i = 1$,

then $G_z = \left\{ gz : g \in G\ ,\ gz = g\sum\limits_{i=1}^{k} m_i\overline{x_i}\ = \sum\limits_{i=1}^{k} m_i\overline{gx_i}\ \right\}$.

Obviously, $G_z = G_{m_1\overline{x_1}+\ldots+m_k\overline{x_k}} = \{m_1\overline{gx_1} + \ldots + m_k\overline{gx_k} :\ g \in G\}$.

Note that $Z(X/G) \cong Z(X)/G$ and is invariant in $Z(X/G)$.

$$\begin{array}{ccc} X \overset{\pi_{G(x)}}{\longrightarrow} X\ /G & & x \to G_x \\ \downarrow i \qquad\qquad \downarrow i & & \\ Z(X)\overrightarrow{\pi_{G(z)}}Z(X)/G & & z \to G_z \end{array}$$

In his paper [9] Cauty proved the following

Lemma 1.2 [1]. Let X be a topological stratified space. If $Y \in S$ and $A \subset Y-$ is closed and $f : A \to X$ is continuous, then the mapping $\tilde{f}$ has a continuous extension $\tilde{f} : Y \to Z(X)$.

Lemma 1. Let X be a topological stratified $G-$ space, $A \subset Y$ a closed $G-$ invariant subset, $f : A \to X$ an equivariant continuous mapping, when the mapping f has a continuous equivariant extension $\tilde{f} : Y \to Z(X)$.

Proof. Let A be a closed subset of the space X. We put $W = X\backslash A$. $W' = \{x \in W : x \in U_y, y \in A$ and U is open in $X\}$ and $m(x) = \max\{n(U, y) : y \in \ A \ and \ x \in U_y\}$. Obviously, $W' \subset W$ and for every $x \in W'$ there is $m(x) < n(W, x) < \infty$.

Let $W = Y\backslash A$, $W' = \{x \in W : x \in U_y, y \in A$ and U is open in $X\}$. Consider the open covered $W^* = \{W_x : x \in W\}$ subspace W. Since the subspace W is paracompact, there exists a locally finite $G-$ cover V inscribed in W^*.

For any $v \in V$, we fix a point (vertex) $x_v \in W$ such that $gx_v = x_{gv}$, where $g \in G$. If a point $x_v \in W'$ we fix such a point (vertex) $a_v \in A$ and an open set $S_v, a_v \in S_v$ such that $x_v \in (S_v)_{a_v}$ and $n(S_v, a_v) = m(x_v)$ and $ga_v = a_{gv}$. If $x_v \overline{\in} W'$ we put a_v fixed $a_0 \in A$. Let $\{P_v : v \in V\}$ be partition of unity subordinate to V and $P_v(x) = P_{gv}(gx)$. The required continuation $F : Y \to Z(X)$ is defined as follows:

$$F(x) = \begin{cases} f(x); & if \ x \in A \\ \sum P_v(x) \cdot f(a_v), & if \ \ x \in W \end{cases}$$

a) Now we show that the mapping $F : Y \to Z(X)$ is equivariant, i.e. $gF(x) = F(gx)$, where $g \in G$.

1) If the point $x \in A$, due to the invariance of the set, we have, $gF(x) = gf(x) = f(gx) = F(gx)$.

2) Let the point $x \in W$ then we have $gF(x) = g\sum(P_v(x) \cdot f(a_v)) = \sum P_v(x) \cdot gf(a_v) = \sum P_v(x)f(a_{gv})$. $F(gx) = \sum P_{gv}g(x) \cdot f(a_{gv}) = \sum P_v(x)f(a_{gv})$.

Hence, $gF(x) = F(gx)$ i.e. the mapping F is equivariant.

b) Due to the -invariance of the closed set A and the simpliciality of a certain mapping $F : Y \to Z(X)$, the mapping $F(x)$ is continuous.

Lemma 1 is proved.

By Lemma 1 and Theorem 1, we have

Theorem 4. The space $X \in G \ A(N)R(S)$ if and only if there is a $G-$retraction r (neighborhood) $G-$ space $Z(X)$ on the $G-$ space X.

Lemma 2. Let $X \in A(N)R(S)$. Then there is a $G-$ retraction $R_n : O(X^n) \to X^n$ such that $G \subseteq S^n$ and $n \in N$ such that $O(X^n)$ is a neighborhood X^n to $Z(X^n)$.

Proof. Let X be $ANR(S)-$ space. It follows from the results of R. Cauty that there is a retraction $r : U \to X$, where U is a neighborhood of the space X in $Z(X)$. We put $V = (r^n)^{-1}(U^n)$, where $V \subset (Z(X))^n$, $r^n : U^n \to X^n$, $U^n \subset (Z(X))^n$.

Now we define the mapping $\varphi : Z(X^n) \to (Z(X))^n$ as follows: $z \in Z(X^n)$, $z = \sum_{i=1}^k m_i\overline{x_i}$, $x_i = (x_1^i, x_2^i, \ldots, \ x_n^i)$. We put $\varphi(z) = (\sum m_i\overline{x_1^i} \sum m_i\overline{x_2^i}, \ \ldots, \ \sum m_i\overline{x_n^i})$. Obviously, $\varphi(z) \in (Z(X))^n$. It is easy to check that the mapping $\varphi : Z(X^n) \to (Z(X))^n$ is continuous. We put $R_n = r^n \ \circ\varphi$ and $\varphi^{-1}(V) = O(\ Z(X^n))$.

Hence $R_n : Z(X^n) \to X^n$. Now we show that R_n is an equivariant mapping, i.e. the equality $R_n(gz) = gR_n(z)$ holds.

$$R_n(gz) = r^n(\varphi(gz)) = r^n(\varphi(\sum m_i\overline{gx_i})) = r^n(\varphi(\sum m_i\overline{g(x_1^i \ \ \ldots, x_n^i)})) =$$
$$= r^n(\varphi(\sum m_i\overline{(x_{g(1)}^i \ \ \ldots, \ x_{g(n)}^i)})) = r^n((\sum m_i \ \overline{x_{g(j)}^i})) = r^n(g(\sum m_i \ \overline{x_j^i})) = g(r^n \sum m_i \ \overline{x_j^i}) = gR_n(z).$$

Hence, the mapping R_n is equivariant. It is easy to check that R_n is a continuous retraction.

Lemma 2 is proved.

Theorem 5. Let $X \in A(N)R(S)$. Then $X^n \in G - A(N)R(S)$, where $G \subseteq S^n$ – is a subgroup of the group of all permutations.

Proof. Let $X \in A(N)R(S)$ and Y be a stratified $G-$space, A its closed invariant $G-$ subset, $f : A \to X^n$ is an arbitrary continuous $G-$mapping. Let $O = \varphi^{-1}(O(Z(X^n)))$. We put $F_g = F{\circ}r^n$, $r^n = R_n^n-$ the Cartesian product of retraction R_n defined by Lemma 2, $F : Y \to Z(X^n)$ mapping defined in Lemma 1.

Then the mapping F_g is a G extension, since F_g is the composition of two G mappings F and r^U. Obviously, F_g is an extension of the mapping f.The theorem is proved.

This theorem implies

Corollary 1. If X is a $G-$ space and $X \in A(N)R(S)$ then $X \in G - A(N)R(S)$.

By virtue of Lemma 1, we can also assert.

Corollary 2. Let X be a topological stratifiable $G-$space. If Y is a stratified $G-$space A is an invariant $G-$space, $f : A \to X|G$ is an equivariant mapping. Then f has an equivariant continuous extension $F : Y \to Z(X/G)$.

Corollary 1 implies.

Corollary 3. If $X \in G - A(N)R(S)$, then $X/G \in G - A(N)R(S)$.

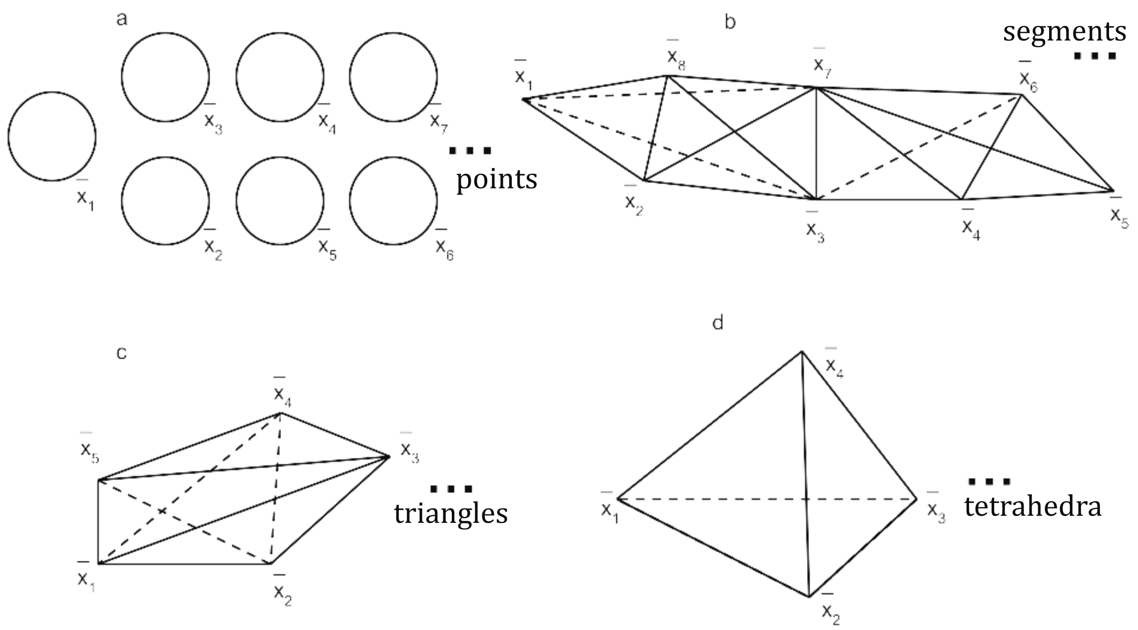

Fig. Illustrations to theorems: a — $z_0(x)$ – x; b — $z_1(x)$ — segments, vertices — point X (one-dimensional simplices); c — $z_2(x)$ — triangles, vertices — points X (two-dimensional simplices; d — $z_3(x)$ — tetrahedra, vertices – points X (three-dimensional simplices)

Рис. Иллюстрации к теоремам: a — $z_0(x)$ — x; b — $z_1(x)$ — отрезки, вершины — точки X (одномерные симплексы); c — $z_2(x)$ — треугольники, вершины — точки X (двумерные симплексы; d — $z_3(x)$ — тетраэдры, вершины — точки X (трехмерные симплексы)

Definition [13]. The set $A \subset X$ is called homotopically dense in X if there exists a homotopy $h(x,t)$: $X \times [0,1] \to X$ such that $h(x,0) = id_X$ and $h(X \times (0,1]) \subset A$.

Theorem 6. For any stratified space X and for any $n \in N^+$ subspace $Z(X) \backslash Z_n(X)$ is homotopically dense in $Z(X)$.

Proof. Let X is the stratified space and $n \in N^+$. Fixing the point $z_0 \in Z(X) \backslash Z_n(X)$, where $z_0 \in <\overline{x}_1, \overline{x}_2, ..., \overline{x}_{n+1}>$ and

$$z_0 = m_1^0 \overline{x}_1 + m_2^0 \overline{x}_2 + ... + m_{n+1}^0 \overline{x}_{n+1},$$

thus supp$z_0 = \{\overline{x}_1, \overline{x}_2, ..., \overline{x}_{n+1}\}$. We will construct the homotopy $h(z,t)$: $Z(X) \times [0,1] \to Z(X)$, assuming $h(z,t) = tz_0 + (1-t)z$.

By virtue of the convexity of the space $Z(X)$ for any $z \in Z(X)$ and $t \in [0,1]$ the point $h(\mu,t)$ belongs to $Z(X)$, that is $h(\mu,t) \in Z(X)$, $\forall z \in Z$ and $\forall t \in [0,1]$.

If $t = 0$, then $h(\mu,0) = z$, that is $h(\mu,0) = id_{Z(X)}$.

If $t > 0$ and $t \leqslant 1$, then $h(\mu,t) = tz_0 + (1-t)z$ belongs to $Z(X) \backslash Z_n(X)$ because the carriers supp$h(\mu,t)$ of point $h(\mu,t)$ consist of at least $n+1$ points, that is

$$h(\mu,t) = tz_0 + (1-t)z =$$

$$= t(m_1^0 \overline{x}_1 + m_2^0 \overline{x}_2 + ... + m_{n+1}^0 \overline{x}_{n+1}) + (1-t)(m_1^0 \overline{x}'_1 + m_2^0 \overline{x}'_2 + ... + m_k^0 \overline{x}'_k) \in Z(X) \backslash Z_n(X),$$

the point $h(\mu,t)$ carrier consists of points z and z_0 carriers, and it consists of different $(n+1)$ points. So the point $h(z,(0,1])\overline{\subset} Z_n(X)$ and $h(z,(0,1]) \subset Z(X) \backslash Z_n(X)$, which was required to be proved. The theorem is proved.

Conclusion

In this paper we consider that the functor Z is open, normal and monadic in the category of stratified spaces and continuous maps to itself. The dimensional properties of the space $Z(X)$ for the stratified space X are also studied, the subfunctor of the functor Z with the corresponding nested dimension is determined for each n.

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ЭКВИВАРИАНТНЫЕ СВОЙСТВА ПРОСТРАНСТВА $\mathbb{Z}(X)$ ДЛЯ СТРАТИФИЦИРУЕМОГО ПРОСТРАНСТВА X

АННОТАЦИЯ

В этой статье доказано, что действие компактной группы G, определяемой стратифицированным пространством X, непрерывно для пространства $Z(X)$, являющегося стратифицированным пространством, содержащим самостратифицированное пространство X как замкнутое подмножество.

Доказан эквивариантный аналог некоторых результатов Р. Коти относительно $A(N)R(S)$-пространств. Также показано, что орбитальное пространство $Z(X)/G$ под действием группы G является пространством S.

Ключевые слова: эквивариантные отображения; стратифицированное пространство; действия группы; орбитальное пространство; инвариантное множество; гомотопическая плотность; размерность; абсолютный экстензор; окрестностный экстензор; ковариантный функтор; вероятностные меры.

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