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NONLINEAR DYNAMIC EQUATIONS FOR ELASTIC MICROMORPHIC SOLIDS AND SHELLS. PART I[1]

ABSTRACT

The present paper develops a general approach to deriving nonlinear equations of motion for solids whose material points possess additional degrees of freedom. The essential characteristic of this approach is the account of incompatible deformations that may occur in the body due to distributed defects or in the result of the some kind of process like growth or remodelling. The mathematical formalism is based on least action principle and Noether symmetries. The peculiarity of such formalism is in formal description of reference shape of the body, which in the case of incompatible deformations has to be regarded either as a continual family of shapes or some shape embedded into non-Euclidean space. Although the general approach yields equations for Cosserat-type solids, micromorphic bodies and shells, the latter differ significantly in the formal description of enhanced geometric structures upon which the action integral has to be defined. Detailed discussion of this disparity is given.

Key words: nonlinear dynamics; micropolar and micromorphic solids; shells; finite deformations; incompatibility of deformations; non-Euclidean reference shape; fiber bundles; enhanced material and physical manifolds; least action; Noether symmetries; field equations; conservation laws.

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Introduction

There is a widespread tendency at present to derive dynamic equations for shell-like solids by direct approach. In doing so, shells are regarded as two-dimensional Cosserat continuum, which additional degrees of freedom are associated with specific shell kinematics. From this point of view, models are similar to three-dimensional micromorphic solids and can be considered within a single mathematical formalism. However the geometric structure of the manifold representing the body has fundamentally different properties, depending on which model is considered. In the language of modern differential geometry, these differences are characterised by a particular bundle structure defined over a manifold that formalizes the shape of the body or its material counterpart. In this paper we have sought to answer on the question: how does the formal structure of bundle relate to the commonly accepted hypotheses in micropolar, micromorphic and shell-like bodies.

Mathematical shell theory is conventional field of continuum mechanics so it is unlikely that a complete literary list can be given here. We will give only some references [1–4]. The mathematical formalization of shell theory in Cosserat framework is also the subject of quite a lot of literature [5–8]. Non-linear models of shells whose deformations do not satisfy the compatibility conditions are much less developed. The works [9; 10] are the first such that should be mentioned here. In them, non-linear models are introduced within a general theory of materially inhomogeneous bodies, primarily developed by Truesdell and Noll in the framework of the material connection theory [11]. This approach has been developed in [12–14]. A slightly different approach based on the modeling of maps between manifolds with dimensions greater than 3 is represented in [15; 16]. Formally, this approach retains the idea of a geometric description for measures of deformation incompatibility in terms of material metric and connection, but defines them in high-dimensional spaces, which correspond to a set of material points, each of which has a continuous set of orientations. Models of this type have specific physical meaning, but it is very different from what is usually found in classical shell models. The latter can be obtained by considering the sections of bundles that form high-dimensional spaces. This issue is discussed in detail in present work.

1. Physical Space and Time

1°. Euclidean physical space and time. Generally, the geometrical approach developed in the paper, regards the body and the physical space as smooth manifolds of the common structure. However, with the application of the developed theory, we will confine ourselves to the specific case of a physical space with Euclidean structure and absolute time, as it is required by classical (non-relativistic) approach. Meanwhile, for the body, as the set of material points, we leave the possibility of being a smooth manifold of a general kind, because the difference between its geometry and Euclidean geometry is precisely that which characterizes the incompatible deformations.

We suppose that physical space can be formalized as three-dimensional Euclidean point space, which can be formulated as the structure[2]

$$\mathbb{E} = (E,\, \mathbb{V},\, \mathrm{vec},\, \cdot,\, \mathrm{or}). \tag{1.1}$$

The first element of the structure represents the underlying set E. Second element of (1.1) is three-dimensional real vector space $\mathbb{V} = (V,\, +_V,\, \cdot_V)$, in which V is the underlying set of translation vectors, $+_V : V \times V \to V$ is operation of addition, and $\cdot_V : \mathbb{R} \times V \to V$ is operation of multiplication on scalars. Points, i. e., elements of E, and translation vectors, i. e., elements of V, are related via the map

$$\mathrm{vec} :\ E \times E \to V, \quad (a,\, b) \mapsto \overrightarrow{ab} := \mathrm{vec}(a,\, b),$$

which belongs to the structure (1.1). The following Weyl axioms are satisfied [17]:

(a) for all three points $a,\, b,\, c \in E$ the Chasles' relation holds:

$$\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac}.$$

[2]One and the same set X can be endowed with various structures, commonly represented by ordered lists of sets and mappings, that turn it into certain mathematical object. Algebraic, topological, smooth and geometric structures may serve as example. If Struct_1 and Struct_2 are structures on the set X then to distinguish between objects, defined by them, we denote these objects as tuples (X, Struct_1) and (X, Struct_2). For example, topology $\mathcal{T}$ and binary operation $\top$ on X define two structures: topological, $(X,\, \mathcal{T})$, and algebraic, $(X,\, \top)$.

(b) for arbitrary point $a \in E$ and arbitrary vector $\mathbf{v} \in V$ there exists a unique point $b \in E$, such that $\overrightarrow{ab} = \mathbf{v}$.

The symbol $(\cdot)$ from (1.1) stands for inner product[3] $\cdot : V \times V \to \mathbb{R}$ on $\mathbb{V}$, which is bilinear, symmetric and positive-definite functional. Finally, or is the chosen orientation of vector space $\mathbb{V}$, i. e., class $[(\boldsymbol{e}_i)_{i=1}^3]$ of bases under the following equivalence relation: bases $(\boldsymbol{e}_i)_{i=1}^3$ and $(\boldsymbol{e}'_i)_{i=1}^3$, $\boldsymbol{e}'_i = \Omega^j{}_i \boldsymbol{e}_i$, are similarly oriented if $\det[\Omega^i{}_j] > 0$. Here $[\Omega^i{}_j]$ is the transition matrix from basis $(\boldsymbol{e}_i)_{i=1}^3$ to basis $(\boldsymbol{e}'_i)_{i=1}^3$. By choosing an orientation, we restrict the set of acceptable bases to the corresponding class.

Having reference standards $(\boldsymbol{i}_k)_{k=1}^3$, one can construct reference standards for area and volume, which are represented by 2-vectors $\boldsymbol{i}_k \wedge \boldsymbol{i}_l$, i. e., oriented unit squares, and 3-vector $\boldsymbol{i}_1 \wedge \boldsymbol{i}_2 \wedge \boldsymbol{i}_3$, i. e., oriented unit cube. Moreover, the choice of inner product and orientation imply isomorphism $* : \mathbb{V} \wedge \mathbb{V} \to \mathbb{V}$, the Hodge star [17,18], from vector space of 2-vectors to vector space $\mathbb{V}$. This allows to introduce cross-product operation

$$[\cdot, \cdot] : \mathbb{V} \times \mathbb{V} \to \mathbb{V}, \quad [\boldsymbol{u}, \boldsymbol{v}] := *(\boldsymbol{u} \wedge \boldsymbol{v}).$$

The absolute time is formalized as one-dimensional manifold $\mathbb{T}$, called chronometric. Its elements are instants of time. Appealing to daily observations, we suppose that $\mathbb{T}$ can be ordered, from "past" to "future". Besides spatial reference standards one has another, temporal reference standard. It defines coordinatization $\mathbb{T} \to \mathbb{R}$ of chronometric manifold $\mathbb{T}$. Assuming that reference standard is fixed, we identify chronometric manifold with real line $\mathbb{R}$.

2°. Enhanced physical space. The above mathematical formalization of the physical space $\mathbb{E}$ contains points which have conventional degrees of freedom only. To describe solids with extra degrees of freedom one needs to enhance the structure of $\mathbb{E}$. Possible way is to associate some m-dimensional manifold $\mathbb{F}$ with each point of $\mathbb{E}$. To formalize this structure in terms of smooth manifolds, further simple reasoning is needed. One can define $(3+m)$-dimensional product manifold $\mathbb{E} \times \mathbb{F}$. The submanifold $\{x\} \times \mathbb{F}$, which is topologically equivalent to $\mathbb{F}$, serves as manifold of possible extra spatial degrees of freedom associated with place $x \in \mathbb{E}$. It is preimage of canonical projection $\mathrm{pr}_{\mathbb{E}} : \mathbb{E} \times \mathbb{F} \to \mathbb{E}$, $\mathrm{pr}_{\mathbb{E}} : (x, f) \to x$, and by this reason, instead of pointing out the collection of all manifolds of the form $\{x\} \times \mathbb{F}$, it is sufficient to specify the projection $\mathrm{pr}_{\mathbb{E}}$. Thus, one arrives at the structure of a bundle $(\mathbb{E} \times \mathbb{F}, \mathbb{E}, \mathrm{pr}_{\mathbb{E}}, \mathbb{F})$. Here $\mathbb{E} \times \mathbb{F}$ is *enhanced physical space* over the manifold of places $\mathbb{E}$, and the manifold $\mathbb{F}$ is collection of extra degrees of freedom. Enhanced physical space and conventional physical space are related via the map $\mathrm{pr}_{\mathbb{E}}$. Note, that here we take extra degrees of freedom from one manifold $\mathbb{F}$, although one may associate with each point $x \in \mathbb{E}$ some manifold $\mathbb{F}_x$, and manifolds $\mathbb{F}_x$ and $\mathbb{F}_y$ for distinct points x and y may differ. Such the case is beyond of this study.

The structure of manifold $\mathbb{F}$ may be rather general. One can consider the following particular cases. In the first case $\mathbb{F}$ is m-dimensional vector space. In the second case we deal with Lie group $(G, \top)$ with group operation $\top$. Elements of G are associated with orientations of points. The structure of Lie group allows to introduce the following map:

$$\lhd_{\mathbb{E}} \colon (\mathbb{E} \times G) \times G \to \mathbb{E} \times G, \quad (x, g) \lhd_{\mathbb{E}} h := (x, g \top h),$$

the action of Lie group G on enhanced physical space $\mathbb{E} \times G$. It defines translations along each fiber $\{x\} \times G$, i. e., one can obtain other orientations of point x from some given one.

We went into detail on the description of these cases because their differences are significant in the derivation of various models for oriented bodies, namely, micropolar, micromorphic and shell-like solids.

2. Material Manifolds

2.1. General Description for Material Manifold

3°. Body manifold. As it was mentioned at the beginning of the study, we leave the possibility for a material manifold to have the most common non-Euclidean geometry. Its formalization therefore calls for more general considerations, presented below. In what follows we use the concepts of a body, its shapes, and mappings between them. All these notions are introduced within the theory of connections on smooth manifolds [19,20]. In particular, the body, i. e., a set of material points, is formalized as a smooth manifold $\mathfrak{B}$ of dimension $n \leqslant 3$ (we refer on it as *body manifold*) [11,21]. This manifold does not carry any metric

[3] Let us dwell into the issue, how to choose inner product $(\cdot)$ and orientation or. Within framework of classical physics it is assumed that one has three rigid reference standards (one for each dimension) and rigid protractor. The reference standards are formalized by fixed three non-coplanar vectors $(\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ from $\mathbb{V}$. Upon measurements it is *defined* that vectors $\boldsymbol{i}_k$ have unit length, mutually orthogonal, and form right-hand triad. From formal viewpoint this means that basis $(\boldsymbol{i}_k)_{k=1}^3$ defines an inner product, with respect to which it is orthogonal: $\boldsymbol{u} \cdot \boldsymbol{v} := \delta_{mn} u^m v^n$ for $\boldsymbol{u} = u^k \boldsymbol{i}_k$ and $\boldsymbol{v} = v^k \boldsymbol{i}_k$; it defines orientation as well: $\mathrm{or} = [(\boldsymbol{i}_k)_{k=1}^3]$.

or connection and defines only topological features of the body. In this regard the body can be viewed as the following structure:

$$\mathfrak{B} = (B,\, \mathcal{T}_B,\, \mathcal{D}_B), \tag{2.1}$$

where B is the underlying set, $\mathcal{T}_B$ is a Hausdorff topology on B, that satisfies the second countability axiom, and $\mathcal{D}_B$ is a smooth structure on the topological manifold $(B, \mathcal{T}_B)$. We denote the points of $\mathfrak{B}$ by uppercase Fraktur symbols like $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$. It should be noted that often a structure for $\mathfrak{B}$ can be derived from the structure for some shape clearing it of Euclidean geometry. This issue will be discussed in detail further, in section 5.2.

4°. Geometry on body manifold. Vividly speaking, the body manifold is a trunk of a tree which branches represent particular spaces. The only question is which branch should be used in modelling. One can obtain the structure of *geometric space* by adding Riemannian metric, or connection, or volume form, or some of their combination, to the basic structure (2.1). Table 1 contains examples of geometric spaces[4], commonly used in geometric continuum mechanics [22–24].

Table 1

Geometric spaces over $\mathfrak{B}$

Space	Structure	Primary fields	Secondary fields
Riemann space	$(B,\, \mathcal{T}_B,\, \mathcal{D}_B,\, \mathsf{g}_B,\, \nabla_B,\, dV_B)$	g_B	$\nabla_B,\, dV_B$
Weitzenböck space	$(B,\, \mathcal{T}_B,\, \mathcal{D}_B,\, \mathsf{g}_B,\, \nabla_B,\, \mu_B)$	$\mathsf{H},\, \mu_B$	$\mathsf{g}_B,\, \nabla_B$
Weyl space	$(B,\, \mathcal{T}_B,\, \mathcal{D}_B,\, \mathsf{g}_B,\, \nabla_B,\, \mu_B)$	$\mathsf{g}_B,\, \nu,\, \mu_B$	∇_B

On Table 1 the field g_B corresponds to Riemann metric, and the field ∇_B corresponds to affine connection. The body manifold $\mathfrak{B}$ is supposed to be oriented [25] and its volume form is denoted by μ_B. The column "Primary fields" contains fields that can be prescribed from some physical reasons that don't depend on the structure of geometry. The latter column, "Secondary fields", contains fields, which can be derived from the primary ones and geometric properties of smooth manifolds. Note that if the space is Riemannian, then the volume form $\mu_B = dV_B$ is determined by metric as $dV_B = \sqrt{\det \mathsf{g}_B}\, d\mathfrak{X}^1 \wedge \dots \wedge d\mathfrak{X}^n$. The connection ∇_B is Levi-Civita connection, being also determined by metric. The affine connection and metric of Weitzenböck space are generated by prescribed field H of linear transformations. Finally, the affine connection of Weyl space can be completely determined by 1-form ν [26].

2.2. Enhanced Material Manifold

5°. Body bundle. In this study we consider bodies, which particles are, in turn, perform themselves as continual sets, consisting of microparticles. With a view to introduce more detailed labelling for constituents of the body, one can endow the body manifold $\mathfrak{B}$ with fiber bundle structure [27,28] $(F\mathfrak{B},\, \mathfrak{B},\, \pi_{\mathfrak{B}},\, \mathbb{F})$. Here $F\mathfrak{B}$ is a smooth manifold, the total space of the bundle, to which we refer as *enhanced body*, $\mathfrak{B}$ is the original manifold of labels of particles, the base of the bundle, $\pi_{\mathfrak{B}} : F\mathfrak{B} \to \mathfrak{B}$ is a smooth surjective map, called projection, and $\mathbb{F}$ is a smooth manifold, called model fiber. Its elements serve as labels for microparticles. Submanifold $F\mathfrak{B}_{\mathfrak{X}} := \pi_{\mathfrak{B}}^{-1}(\{\mathfrak{X}\})$ is the fiber over point $\mathfrak{X}$. To stay in conventional framework of differential geometry, the following condition is required to be hold: for each point $\mathfrak{X} \in \mathfrak{B}$ there exist a neighborhood U in $\mathfrak{B}$ and a diffeomorphism $\Phi : \pi_{\mathfrak{B}}^{-1}(U) \to U \times \mathbb{F}$, called local trivialization, such that

$$\mathrm{pr}_U \circ \Phi = \pi_{\mathfrak{B}}, \tag{2.2}$$

where $\mathrm{pr}_U : U \times \mathbb{F} \to U$ is the canonical projection of the Cartesian product onto the first factor. Thus, diffeomorphism Φ has the following representation: $\Phi(p) = (\pi_{\mathfrak{B}}(p),\, \varphi(p))$, for some smooth mapping $\varphi : \pi_{\mathfrak{B}}^{-1}(U) \to \mathbb{F}$.

The local trivialization property means that enhanced body $F\mathfrak{B}$ is locally arranged as Cartesian product of some part of $\mathfrak{B}$ and the model fiber $\mathbb{F}$. This allows one to introduce special local coordinates on $F\mathfrak{B}$. Indeed, let $m = \dim \mathbb{F}$, and let $\Phi : \pi_{\mathfrak{B}}^{-1}(U) \to U \times \mathbb{F}$ be local trivialization, where U is also coordinate domain of some chart $(U,\, \sigma_{\mathfrak{B}})$ on $\mathfrak{B}$; here $\sigma_{\mathfrak{B}} : U \to \mathbb{R}^n$ is coordinate map. Moreover, let $(V,\, \sigma_{\mathbb{F}})$ be a chart on $\mathbb{F}$; here $\sigma_{\mathbb{F}} : V \to \mathbb{R}^m$ is also coordinate map. Then put $O := \Phi^{-1}(U \times V) \subset \pi_{\mathfrak{B}}^{-1}(U)$. The restriction

$$\Phi_O = \Phi|_O :\, O \to U \times V$$

is homeomorphism and by this reason[5]

$$\Sigma = (\sigma_{\mathfrak{B}} \times \sigma_{\mathbb{F}}) \circ \Phi_O :\, O \to \mathbb{R}^n \times \mathbb{R}^m, \quad \Sigma(p) = (\mathfrak{X}^1,\, \dots,\, \mathfrak{X}^n;\, \mathfrak{f}^1, \dots,\, \mathfrak{f}^m),$$

[4]More detailed description of what metric and connection give to body manifold can be found in paragraph **23°** of the Appendix.

[5]Here $f_1 \times f_2 :\, X_1 \times X_1 \to Y_1 \times Y_2$ is designation for product of mappings $f_i :\, X_i \to Y_i$, $i = 1,\, 2$, defined as $f_1 \times f_2(x_1,\, x_2) := (f_1(x_1),\, f_2(x_2))$.

is coordinate homeomorphism. Thus, (O, Σ) is a chart on enhanced body $F\mathfrak{B}$. We refer to coordinates, induced by this chart, as natural coordinates.

The chart (O, Σ) has the property, that any point p from O can be completely determined by coordinates of the base manifold $\mathfrak{B}$ and coordinates of the model fiber $\mathbb{F}$. Moreover, the coordinate representation

$$\widetilde{\pi}_{\mathfrak{B}} = \sigma_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \Sigma^{-1} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

of the projection $\pi_{\mathfrak{B}}$ coincides with the canonical projection of $\mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$.

The structure of general fiber bundle is very flexible. One can obtain particular cases by assuming that model fiber $\mathbb{F}$ is either vector space or Lie group and imposing the corresponding restrictions on local trivializations. Due to importance of these cases in our study, we discuss them individually. Examples of bundles, listed below, would be used in various theories of micropolar fields considered in the study.

6°. The case of vector bundle. Material vector bundle of rank m [19] is the fiber bundle $(V\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, \mathbb{F})$ over $\mathfrak{B}$, where the substructure $\mathbb{F}$ is m-dimensional real vector space, which elements are orientations of the medium. It is supposed that for every $\mathfrak{X} \in \mathfrak{B}$ the preimage $\pi_{\mathfrak{B}}^{-1}(\{\mathfrak{X}\}) =: V\mathfrak{B}_{\mathfrak{X}}$ is m-dimensional real vector space. Moreover, local trivialization $\Phi : \pi_{\mathfrak{B}}^{-1}(U) \to U \times \mathbb{F}$ satisfies, beyond the property (2.2), the following requirement: for all $\mathfrak{Y} \in U$, the mapping $\Phi|_{V\mathfrak{B}_{\mathfrak{Y}}} : V\mathfrak{B}_{\mathfrak{Y}} \to \mathbb{F}$ is isomorphism between vector spaces $V\mathfrak{B}_{\mathfrak{Y}}$ and $\{\mathfrak{Y}\} \times \mathbb{F} \cong \mathbb{F}$.

Thus, like in the case of general fiber bundle, the whole enhanced body $V\mathfrak{B}$ is split into disjoint union $V\mathfrak{B} = \coprod_{x\in\mathfrak{B}} V\mathfrak{B}_{\mathfrak{X}}$ of fibers $V\mathfrak{B}_{\mathfrak{X}}$, which contain all possible material orientations of particle $\mathfrak{X}$. Since fibers $V\mathfrak{B}_{\mathfrak{X}}$ are linearly isomorphic to typical fiber $\mathbb{F}$, i. e., $V\mathfrak{B}_{\mathfrak{X}} \cong \mathbb{F}$, in reasonings one can replace them by elements of the model fiber.

The bundle structure allows one to introduce natural coordinates on $V\mathfrak{B}$. Choose some coordinates $(\mathfrak{X}^i)_{i=1}^n$ on the base manifold $\mathfrak{B}$ and choose some basis $(e_i)_{i=1}^m$ for $\mathbb{F}$. Then every point of $V\mathfrak{B}$ can be completely characterized by $(m+n)$-tuple $(\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{v}^1, ..., \mathfrak{v}^m) \in \mathbb{R}^{m+n}$.

7°. The case of principal bundle. Material principal bundle with structure group G [29] is a structure $(P\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, G, \top, \lhd_{\mathfrak{B}})$, in which $P\mathfrak{B}$, $\mathfrak{B}$, and $\pi_{\mathfrak{B}} : P\mathfrak{B} \to \mathfrak{B}$ are, as earlier, enhanced body, body manifold, and projection. The substructure $(G, \top)$ is Lie group with binary operation $\top$, and $\lhd_{\mathfrak{B}}$: $P\mathfrak{B} \times G \to P\mathfrak{B}$ is a smooth right action of the group G on manifold $P\mathfrak{B}$. The following properties are assumed:

(1) The action preserves fibers of $\pi_{\mathfrak{B}}$:

$$\forall p \in P\mathfrak{B}\, \forall g \in G: \ \pi_{\mathfrak{B}}(p \lhd_{\mathfrak{B}} g) = \pi_{\mathfrak{B}}(p).$$

(2) For every point $\mathfrak{X} \in \mathfrak{B}$ there exists a neighborhood U in $\mathfrak{B}$ and a diffeomorphism $\Phi : \pi_{\mathfrak{B}}^{-1}(U) \to U \times G$, such that $\Phi(p) = (\pi_{\mathfrak{B}}(p), \varphi(p))$, where the mapping $\varphi : \pi_{\mathfrak{B}}^{-1}(U) \to G$ is equivariant, that is,

$$\forall p \in \pi_{\mathfrak{B}}^{-1}(U)\, \forall g \in G: \ \varphi(p \lhd_{\mathfrak{B}} g) = \varphi(p) \top g.$$

Thus, material principal bundle $(P\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, G, \top, \lhd_{\mathfrak{B}})$ is particular case of material fiber bundle $(F\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, \mathbb{F})$. The typical fiber is “linear” in the sense that it is represented by Lie group $(G, \top)$, and Lie group structure is aligned with fiber bundle structure through group action. In particular, manifold $P\mathfrak{B}$ can be endowed with natural coordinates $(\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{g}^1, ..., \mathfrak{g}^m)$. Here m is dimension of G. With respect to natural coordinates projection, as well as group action, can be represented in laconic form. Like for general case of fiber bundle, the coordinate representation of projection $\pi_{\mathfrak{B}} : P\mathfrak{B} \to \mathfrak{B}$ is canonical projection:

$$\widetilde{\pi}_{\mathfrak{B}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \quad \widetilde{\pi}_{\mathfrak{B}}(\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{g}^1, ..., \mathfrak{g}^m) = (\mathfrak{X}^1, ..., \mathfrak{X}^n).$$

The coordinate representation of action $\lhd_{\mathfrak{B}}$ is the map

$$(\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{g}^1, ..., \mathfrak{g}^m)\widetilde{\lhd}_{\mathfrak{B}}(\mathfrak{h}^1, ..., \mathfrak{h}^m) = (\mathfrak{X}^1, ..., \mathfrak{X}^n; (\mathfrak{g}^1, ..., \mathfrak{g}^m)\widetilde{\top}(\mathfrak{h}^1, ..., \mathfrak{h}^m)),$$

where $\widetilde{\top} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is coordinate representation of group operation $\top$.

8°. Frame bundle. If general linear group $\mathrm{GL}(n; \mathbb{R})$ is chosen as structure group, then one arrives at material frame bundle $(L\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, \mathrm{GL}(n; \mathbb{R}), ., \lhd_{\mathfrak{B}})$, in which total space is manifold over the set

$$L\mathfrak{B} = \coprod_{\mathfrak{X}\in\mathfrak{B}} L_{\mathfrak{X}}\mathfrak{B}, \quad L_{\mathfrak{X}}\mathfrak{B} = \{(\mathsf{e}_i)_{i=1}^n \mid (\mathsf{e}_i)_{i=1}^n \text{ is a basis for } T_{\mathfrak{X}}\mathfrak{B}\},$$

the projection $\pi_{\mathfrak{B}} : L\mathfrak{B} \to \mathfrak{B}$ is defined as

$$\pi_{\mathfrak{B}}(\mathfrak{X}, (\mathsf{e}_i)_{i=1}^n) := \mathfrak{X},$$

and action $\lhd_{\mathfrak{B}}$: $L\mathfrak{B} \times \mathrm{GL}(n; \mathbb{R}) \to L\mathfrak{B}$ is

$$(\mathfrak{X}, (\mathsf{e}_i)_{i=1}^n) \lhd_{\mathfrak{B}} [\Omega^i{}_j] := (\mathfrak{X}, (\Omega^j{}_i \mathsf{e}_j)_{i=1}^n).$$

Here (.) denotes matrix product. Local trivialization Φ can be defined as follows. Choose some chart (U, σ) on body manifold $\mathfrak{B}$ and let $(\partial_i)_{i=1}^n$ be coordinate frame with respect to this chart. Then for any basis $(\mathsf{e}_i)_{i=1}^n \in L_{\mathfrak{X}}\mathfrak{B}$, $\mathfrak{X} \in \mathfrak{B}$, one has decomposition $\mathsf{e}_i = {}^\sigma\Omega^j{}_i \partial_j|_{\mathfrak{X}}$, where $[{}^\sigma\Omega^i{}_j] \in \mathrm{GL}(n;\, \mathbb{R})$. Introduce the mapping $\Phi:\, \pi_{\mathfrak{B}}^{-1}(U) \to U \times \mathrm{GL}(n;\, \mathbb{R})$ by equality

$$\Phi(\mathfrak{X},\, (\mathsf{e}_i)_{i=1}^n) := (\mathfrak{X},\, [{}^\sigma\Omega^i{}_j]).$$

In particular, $\varphi:\, \pi_{\mathfrak{B}}^{-1}(U) \to \mathrm{GL}(n; \mathbb{R})$ acts as $\varphi(\mathfrak{X},\, (\mathsf{e}_i)_{i=1}^n) = [{}^\sigma\Omega^i{}_j]$, which implies the equivariance property. The dimension of enhanced body $L\mathfrak{B}$ is $\dim L\mathfrak{B} = n + n^2$, and the representation on natural coordinates is of the form

$$L\mathfrak{B} \ni p \mapsto (\mathfrak{X}^1,\, ...,\, \mathfrak{X}^n;\, \mathfrak{g}^1{}_1,\, ...,\, \mathfrak{g}^n{}_n) \in \mathbb{R}^n \times \mathbb{R}^{n^2}.$$

3. Kinematic Description

3.1. Conventional Kinematics

9°. Conventional configurations and deformations. Body manifold is not directly observable to us, inhabitants of three-dimensional Euclidean space $\mathbb{E}$. One can observe shapes, that are the images of configurations. By a *configuration* [11] we mean a smooth embedding[6] $\varkappa:\ \mathfrak{B} \to \mathbb{E}$. Thus, we formalize shapes as images $\mathbb{S}_\varkappa = \varkappa(\mathfrak{B})$. Note, that the image $\mathbb{S}_\varkappa$ may not coincide with the whole physical space $\mathbb{E}$, even if $\mathfrak{B}$ has three dimensions. By this reason, $\varkappa$ is not invertible. To remedy the issue, one can define a new map $\widehat{\varkappa}:\, \mathfrak{B} \to \mathbb{S}_\varkappa$, such that $\widehat{\varkappa}(\mathfrak{X}) = \varkappa(\mathfrak{X})$. Then the new map is invertible, as desired.

Let $\varkappa_R,\, \varkappa:\ \mathfrak{B} \to \mathbb{E}$ be configurations which images are shapes $\mathbb{S}_R$ and $\mathbb{S}$ respectively. The change of shapes is characterized by mapping $\gamma := \widehat{\varkappa} \circ \widehat{\varkappa}_R^{-1}:\ \mathbb{S}_R \to \mathbb{S}$, to which we refer as *deformation*. Relations between the configurations and deformations are illustrated on Fig. 1.

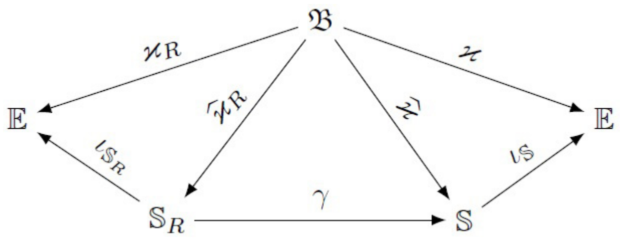

Fig. 1. Relations between the configurations and deformations

On Fig. 1, $\mathbb{S}_R = \varkappa_R(\mathfrak{B})$ and $\mathbb{S} = \varkappa(\mathfrak{B})$. The assignments $\iota_{\mathbb{S}_R}:\ \mathbb{S}_R \hookrightarrow \mathbb{E}$ and $\iota_{\mathbb{S}}:\ \mathbb{S} \hookrightarrow \mathbb{E}$ are inclusion maps[7].

10°. Motion. Suppose that we have fixed some reference shape $\mathbb{S}_R$. We define motion of the body as a family $\{\gamma_t\}_{t\in T}$ of deformations $\gamma_t:\ \mathbb{S}_R \to \mathbb{S}_t$, which index set $T \subset \mathbb{R}$ is an interval. This is conventional viewpoint, presented in such monographs as [31]. It is also convenient to represent the motion in terms of one smooth mapping $\gamma:\, \mathbb{S}_R \times T \to \mathbb{E}$ defined as $\gamma(X,\, t) := \gamma_t(X)$.

Another viewpoint on motion uses family of shapes instead of one global shape. The need of such generalization arises in the case, when body doesn't have reference shape with desired properties (in particular, stress-free), but there is a family of shapes, locally satisfying the required properties. This viewpoint is discussed in detail in section 5.1.

It is appropriate to note about the following possibility for position definition in physical space. Affine structure of physical space $\mathbb{E}$ allows to describe points through translation vectors ("vectorization"). To do this, one chooses some origin $o \in \mathbb{E}$ and defines invertible field $\mathbf{p}:\, \mathbb{E} \to \mathbb{V}$ of radius-vectors $\mathbf{p}(x) := \overrightarrow{ox}$. If we denote radius-vectors of points X from reference shape $\mathbb{S}_R$ by $\mathbf{X}$, i. e., $\mathbf{X} = \mathbf{p}(X)$, then we can represent mapping γ in vectorial form:

$$\boldsymbol{\chi}(\mathbf{X},\, t) := \mathbf{p}[\gamma(\mathbf{p}^{-1}(\mathbf{X}),\, t)].$$

[6]A smooth embedding from one smooth manifold M to another smooth manifold N is a mapping $\varkappa:\, M \to N$, which is 1) smooth, i. e., its coordinate representation is smooth in the sense of conventional Calculus; 2) at every point of M the rank of the Jacobi matrix formed upon coordinate representation of $\varkappa$ is equal to $\dim M$; 3) $\varkappa$ is a homeomorphism onto its image [25]. By requiring from configuration to be an embedding we exclude undesired situations like self-intersections.

[7]Apart formal mathematical need to use inclusion maps, there is a physical need as well. Suppose that body manifold is two-dimensional. Then one can describe its stress-strain state by means of inner geometry, i. e., using two-dimensional vectors, tensors, etc. [30]. Meanwhile, to relate fields of strains and stresses with their three-dimensional counterparts, one needs to use inclusion map. Contact of three-dimensional body with two-dimensional membrane may serve as example of such the problem.

The mapping $\boldsymbol{\chi} : \mathbb{V}_R \times T \to \mathbb{V}$, where $\mathbb{V}_R := \mathbf{p}(\mathbb{S}_R)$, is used in further considerations as well. Note the distinction between mappings γ and $\boldsymbol{\chi}$. The former mapping transforms points to points and its definition doesn't require Euclidean structure of the ambient manifold, i. e., it can be generalized to arbitrary manifolds. Meanwhile, the latter mapping transforms radius-vectors to radius-vectors an by this reason exists in Euclidean manifold only.

3.2. Enhanced Kinematics

11°. Enhanced configurations and deformations. In a manner similar to body manifold we suppose that physical manifold $\mathbb{E}$ is endowed with fiber bundle structure $(F\mathbb{E}, \mathbb{E}, \pi_{\mathbb{E}}, \mathbb{F})$ with the same model fiber $\mathbb{F}$ as for the body (in **2°** we have considered trivial bundle; here we suppose that this bundle may be of general form). Total space of the bundle $F\mathbb{E}$ plays role of enhanced physical space. Its k-dimensional submanifolds (here $k = \dim F\mathfrak{B}$) generalize conventional shapes of the body; thus, shape of the body is not only region of places, but also a collection of extra degrees of freedom.

Following the general methodology of continuum mechanics, we consider configurations and deformations as embeddings of one smooth manifold into another smooth manifold. By this reason, we define enhanced configuration as smooth embedding [15, 16] $F\varkappa : F\mathfrak{B} \to F\mathbb{E}$ of enhanced body $F\mathfrak{B}$ to enhanced physical space $F\mathbb{E}$. We suppose that this mapping preserves fibers, i. e., there exists an embedding $\varkappa : \mathfrak{B} \to \mathbb{E}$ of the body manifold to the physical space, the conventional configuration, such that

$$\varkappa \circ \pi_{\mathfrak{B}} = \pi_{\mathbb{E}} \circ F\varkappa. \tag{3.1}$$

If one introduces natural coordinates $(\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{f}^1, ..., \mathfrak{f}^m)$ on enhanced body and natural coordinates $(x^1, x^2, x^3; f^1, ..., f^m)$ on enhanced physical space, then coordinate representation of $F\varkappa$ would have the form

$$\widetilde{F\varkappa}(\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{f}^1, ..., \mathfrak{f}^m) = (\widetilde{\varkappa}(\mathfrak{X}^1, ..., \mathfrak{X}^n); \tau(\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{f}^1, ..., \mathfrak{f}^m)),$$

where $\widetilde{\varkappa} : (\mathfrak{X}^1, ..., \mathfrak{X}^n) \mapsto (x^1, x^2, x^3)$ is a coordinate representation of configuration $\varkappa$, and $\tau : (\mathfrak{X}^1, ..., \mathfrak{X}^n; \mathfrak{f}^1, ..., \mathfrak{f}^m) \mapsto (f^1, ..., f^m)$ characterizes those part of embedding (in coordinate representation), which with material degrees of freedom associates their spatial counterpart.

Let $F\varkappa_R, F\varkappa : F\mathfrak{B} \to F\mathbb{E}$ be enhanced configurations, to which correspond conventional configurations $\varkappa_R, \varkappa : \mathfrak{B} \to \mathbb{E}$ via (3.1). Then their images $F\mathbb{S}_R = F\varkappa_R(F\mathfrak{B})$ and $F\mathbb{S} = F\varkappa(F\mathfrak{B})$ correspond to shapes of enhanced material manifold in enhanced physical space. Each of them is smooth submanifold of $F\mathbb{E}$ and one can restrict fiber bundle structure from $F\mathbb{E}$ to these shapes, i. e., to consider fiber bundles

$$(F\mathbb{S}_R, \mathbb{S}_R, \pi_{\mathbb{E}}|_{F\mathbb{S}_R}, \mathbb{F}), \quad (F\mathbb{S}, \mathbb{S}, \pi_{\mathbb{E}}|_{F\mathbb{S}}, \mathbb{F}),$$

where $\mathbb{S}_R = \varkappa_R(\mathfrak{B})$, $\mathbb{S} = \varkappa(\mathfrak{B})$ are n-dimensional submanifolds of physical space $\mathbb{E}$, that represent conventional shapes of the body. Of course, they are, in some sense, projections of enhanced shapes $F\mathbb{S}_R$ and $F\mathbb{S}$, and by this reason the information about extra variables is lost. Mappings $\pi_{\mathbb{E}}|_{F\mathbb{S}_R} : F\mathbb{S}_R \to F\mathbb{S}_R$ and $\pi_{\mathbb{E}}|_{F\mathbb{S}} : F\mathbb{S} \to F\mathbb{S}$ are restrictions of the projection $\pi_{\mathbb{E}}$ to the corresponding shapes.

Bearing in mind the intuitive meaning of deformation as change of shapes we define composition[8]

$$F\gamma = \widehat{F\varkappa} \circ \widehat{F\varkappa_R}^{-1} : F\mathbb{S}_R \to F\mathbb{S},$$

to which we refer as enhanced deformation. If $\varkappa_R, \varkappa : \mathfrak{B} \to \mathbb{E}$ are conventional configurations that correspond to enhanced ones, then the composition $\gamma = \widehat{\varkappa} \circ \widehat{\varkappa}_R^{-1} : \mathbb{S}_R \to \mathbb{S}$ is the conventional deformation, i. e., change of places. This is exactly what one observes in physical space if not using experimental setup that allows to identify extra degrees of freedom. Enhanced deformation $F\gamma$ and conventional deformation γ are related by formula analogous to (3.1):

$$\gamma \circ \pi_{\mathbb{E}}|_{F\mathbb{S}_R} = \pi_{\mathbb{E}}|_{F\mathbb{S}} \circ F\gamma. \tag{3.2}$$

Relations between shapes, configurations, deformations and their enhanced counterparts are shown on Fig. 2.

For coordinate representation of enhanced deformation choose natural coordinates[9] $(X^1, ..., X^n; F^1, ..., F^m)$ on reference shape $F\mathbb{S}_R$ and natural coordinates $(x^1, ..., x^n; f^1, ..., f^m)$ on actual shape $F\mathbb{S}$. Then coordinate representation of deformation $F\gamma$ has the form

$$\widetilde{F\gamma}(X^1, ..., X^n; F^1, ..., F^m) = (\widetilde{\gamma}(X^1, ..., X^n); \widetilde{\varepsilon}(X^1, ..., X^n; F^1, ..., F^m)).$$

[8]Like in the case of conventional configurations (see **9°**), the mappings $F\varkappa_R$ and $F\varkappa$ are not invertible in general. To define deformation as change of shape, we restrict the codomains of these mappings to the images. So obtained maps are denoted with hat signs.

[9]In these tuples, $(X^1, ..., X^n)$ are coordinates on the shape $\mathbb{S}_R$ and $(x^1, ..., x^n)$ are coordinates on the shape $\mathbb{S}$; both shapes are considered as manifolds on its own right.

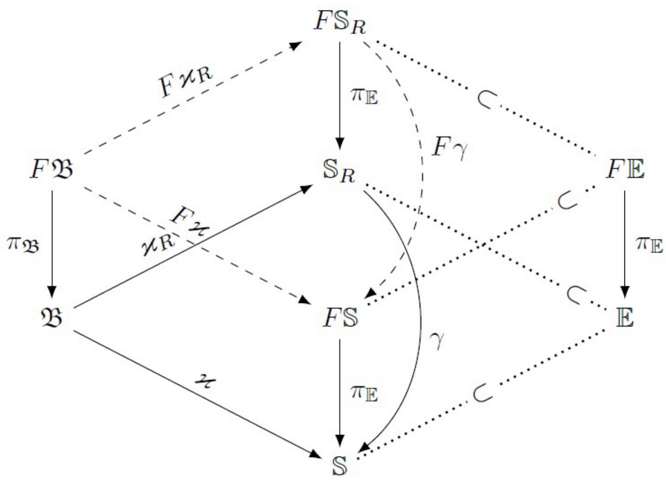

Fig. 2. Enhanced configurations and deformations

Here $\widetilde{\gamma}\ :\ (X^1,\ ...,\ X^n)\ \mapsto\ (x^1,\ ...,\ x^n)$ is coordinate representation of deformation γ, and $\widetilde{\varepsilon}(X^1, ..., X^n; F^1, ..., F^m) \mapsto (f^1, ..., f^m)$ is the mapping, which assigns with reference values of extra degrees of freedom their actual values.

Note, that in general, mapping $\widetilde{\varepsilon} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ doesn't have counterpart among "point to point" mappings $F\mathbb{S}_R \to \mathbb{F}$. Meanwhile, there is a particular case, when it has such the counterpart. Suppose that the physical bundle $(F\mathbb{E},\ \mathbb{E},\ \pi_{\mathbb{E}},\ \mathbb{F})$ is trivial, i. e., local trivialization $\Phi :\ \pi_{\mathbb{E}}^{-1}(U) \to U \times \mathbb{F}$ is global: $U = \mathbb{E}$. Then one can replace $F\mathbb{E}$ by $\mathbb{E} \times \mathbb{F}$ and $\pi_{\mathbb{E}}$ by canonical projection $\mathrm{pr}_{\mathbb{E}} :\ \mathbb{E} \times \mathbb{F} \to \mathbb{E}$. By this reason, one can consider enhanced deformation as mapping $F\gamma :\ \mathbb{S}_R \times \mathbb{F} \to \mathbb{S} \times \mathbb{F}$, and the condition (3.2) implies, that there exists a mapping $\varepsilon :\ \mathbb{S}_R \times \mathbb{F} \to \mathbb{F}$, such that

$$F\gamma(X,\ F) = (\gamma(X),\ \varepsilon(X,\ F)). \tag{3.3}$$

One sees, that in this case deformation of solids with extra degrees of freedom is characterized by two mappings $\gamma : \mathbb{S}_R \to \mathbb{S}$ and $\varepsilon : \mathbb{S}_R \times \mathbb{F} \to \mathbb{F}$, which, in natural coordinates, have representations $\widetilde{\gamma}$ and $\widetilde{\varepsilon}$. Defined above enhanced configurations and deformations are mathematically close to the generalized deformations introduced in [15].

12°. Sections. Mathematical structure based on the idea of embedding of an enhanced material manifold into an enhanced physical space is rather general. In particular, it allows for continual set of possible orientations, associated with a material point. To derive conventional model of oriented solid, one has to limit this set in a certain way. One possible approach is to define a section over the bundle, thereby choose a single reference orientation for each material point.

Let $(F\mathfrak{B},\ \mathfrak{B},\ \pi_{\mathfrak{B}},\ \mathbb{F})$ be material body bundle. A smooth section of this bundle is a smooth mapping $\sigma : \mathfrak{B} \to F\mathfrak{B}$, such that $\pi_{\mathfrak{B}} \circ \sigma = \mathrm{Id}_{\mathfrak{B}}$. The latter equality means that $\sigma(\mathfrak{X}) \in F\mathfrak{B}_{\mathfrak{X}}$ for each $\mathfrak{X} \in \mathfrak{B}$. Choosing section σ, we arrive at the structure $(F\mathfrak{B},\ \mathfrak{B},\ \pi_{\mathfrak{B}},\ \mathbb{F},\ \sigma)$, to which we refer as oriented solid. In what follows we suppose that σ is an embedding. Then configuration of oriented solid is formalized as an embedding

$$F\varkappa_\sigma := F\varkappa \circ \sigma :\ \mathfrak{B} \to F\mathbb{E},$$

where $F\varkappa :\ F\mathfrak{B} \to F\mathbb{E}$ is some enhanced configuration. With regard to deformations, we obtain that the equality (3.3) takes the form

$$F\gamma_U(X) = (\gamma(X),\ \varepsilon(X,\ U_X)),$$

where γ is conventional deformation, while $\mathbb{S}_R \ni X \mapsto U_X \in \mathbb{F}$ is section of fiber bundle $(F\mathbb{S}_R, \mathbb{S}_R, \pi_{\mathbb{E}}|_{F\mathbb{S}_R}, \mathbb{F})$, induced by section σ. Thus, we arrive at the following representation for deformation of micromorphic solid:

$$F\gamma_U(X) = (\gamma(X),\ \widehat{\varepsilon}(X)),$$

and it means that deformation is completely characterized by mappings $\gamma :\ \mathbb{S}_R \to \mathbb{S}$ and $\widehat{\varepsilon} :\ \mathbb{S}_R \to \mathbb{F}$.

Similarly, when one deals with material vector bundle (see paragraph **24°** of the Appendix), the formula (5.10) for enhanced deformation reduces to

$$V\gamma_U(X) = (\gamma(X),\ \boldsymbol{\mathcal{M}}(X,\ U_X)),$$

or, denoting $\boldsymbol{\mathcal{M}}_U(X) := \boldsymbol{\mathcal{M}}(X,\ U_X)$, to

$$V\gamma_U(X) = (\gamma(X),\ \boldsymbol{\mathcal{M}}_U(X)). \tag{3.4}$$

Here $X \mapsto \boldsymbol{\mathcal{M}}_U(X)$ is field $\mathbb{S}_R \to \mathbb{F}$ of actual directors.

There is an issue related with the above considerations. As it is assumed, with each point X of reference shape $\mathbb{S}_R$ one associates some vector $\boldsymbol{d}_X^R$ from $\mathbb{F}$. A deformation causes change of shape and then particles of body occupy some shape $\mathbb{S}$. Orientations $\boldsymbol{d}_X^R$, in turn, deform to orientations $\boldsymbol{d}_x$, associated with points x from $\mathbb{S}$. Meanwhile, this picture depends on observer: different observers may choose different reference orientations, which can transform to different actual orientations as well. To make considerations objective, one needs to consider relations between orientations instead of the orientations themselves. But these relations are hidden in (5.10). Indeed, for each $X \in \mathbb{S}_R$ one has linear partial mapping[10] $\boldsymbol{\mathcal{M}}_X \in \mathrm{Lin}(\mathbb{F};\,\mathbb{F})$. This mapping transforms reference orientation U to actual orientation $u = \boldsymbol{\mathcal{M}}_X[U]$. If we deal with sections, then formula (3.4) should be replaced by

$$V\gamma_U(X) = (\gamma(X),\, \boldsymbol{\mathcal{M}}_X[U_X]). \tag{3.5}$$

Here $U : X \mapsto U_X$ is fixed section on reference shape, i. e., field of reference directors. Thus, in fact, deformation is completely characterized by mapping $\gamma : \mathbb{S}_R \to \mathbb{S}$, the conventional deformation, and field $\boldsymbol{\mathcal{M}} : \mathbb{S}_R \to \mathrm{Lin}(\mathbb{F};\,\mathbb{F})$ of linear maps, which relate reference and actual orientations.

Note, that since shapes $\mathbb{S}_R$ and $\mathbb{S}$ are equitable, the enhanced deformation $V\gamma$ is a diffeomorphism. This implies, in particular, that linear map $\boldsymbol{\mathcal{M}}_X$ from formula (3.5) is invertible, as well as conventional deformation γ.

4. Action Integral

4.1. Action Integral of Conventional Elasticity

13°. Classical Lagrangian. The main focus of this study is the action integral, which will be denoted with the symbol $\mathscr{J}$. We believe that $\mathscr{J}$ can be defined as the integral of a smooth scalar function of a certain set of independent and dependent variables (fields). All follow-up analysis depends upon the choice of these variables and their geometric structure (scalar, vectorial or tensorial).

For classical (non-polar) continuous medium action integral can be written in the following form:

$$\mathscr{J} = \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{S}_R} \mathscr{L}\, dV\, dt, \quad \mathscr{L} = \mathscr{L}(\mathbf{X}, t, \boldsymbol{\chi}, \boldsymbol{\nabla\chi}, \dot{\boldsymbol{\chi}}), \tag{4.1}$$

where the integration is performed over the arbitrary time interval $[t_1,\, t_2]$ and arbitrary reference domain $\mathbb{S}_R \subset \mathbb{E}$. In this expression $\mathscr{L}$ is a Lagrangian density per unit of the volume of reference shape, $\mathbf{X}$ is the position of material particle in reference shape identified with radius-vector, t is a time variable, $\boldsymbol{\chi} = \boldsymbol{\chi}(\mathbf{X}, t)$ represents positions of material particles in actual shape, and $\boldsymbol{\nabla\chi}$ is position gradient[11].

It should be noted that, generally, the dimension of $\mathbb{S}_R$ may be less than for $\mathbb{E}$. Then instead of using three-dimensional field $\boldsymbol{\nabla\chi}$ one needs to use two-dimensional surface gradient $\boldsymbol{\nabla}_s\boldsymbol{\chi}$ introduced in [30]. This will play a significant role in the derivations for shell-like bodies. Above all, let us make no assumption about the property of reference shape to be free from stresses.

4.2. Solids with Orientations

14°. Preliminary reasonings. Even in the framework of conventional elasticity one has to take into account that every deformation within infinitesimal neighborhood of any point X from reference shape has "rotation" part. During deformation each infinitesimal fiber emanating from X stretches and rotates according to Cauchy's polar decomposition theorem [18]: deformation gradient $\boldsymbol{\nabla\chi}$ can be represented as the product of an orthogonal tensor $\boldsymbol{O}$ and symmetric tensor $\boldsymbol{V}$, i. e.,

$$\boldsymbol{\nabla\chi} = \boldsymbol{V}\!\cdot\!\boldsymbol{O}, \quad \boldsymbol{O}^{\mathrm{T}} = \boldsymbol{O}^{-1}, \quad \boldsymbol{V} = \boldsymbol{V}^{\mathrm{T}} = \left(\boldsymbol{\nabla\chi}\!\cdot\!(\boldsymbol{\nabla\chi})^{\mathrm{T}}\right)^{1/2}. \tag{4.2}$$

The first equality of (4.2) implies that the tensor field $\boldsymbol{O}$ is related with actual positions $\boldsymbol{\chi}$ of particles:

$$\boldsymbol{O} = \boldsymbol{V}^{-1}\!\cdot\!\boldsymbol{\nabla\chi} = \left(\boldsymbol{\nabla\chi}\!\cdot\!(\boldsymbol{\nabla\chi})^{\mathrm{T}}\right)^{-1/2}\!\cdot\!\boldsymbol{\nabla\chi}. \tag{4.3}$$

Formula (4.3) shows that rotations of infinitesimal fibers are embodied in the deformation gradient and are not independent from $\boldsymbol{\chi}$: elementary volume, associated with $\mathbf{X}$, stretches and rotates as the whole; after

[10] Hereafter the symbol $\mathrm{Lin}(U;\,V)$ denotes vector space of linear maps $U \to V$.

[11] Here $\boldsymbol{\nabla\chi}$ is considered as indivisible symbol; $\boldsymbol{\nabla\chi} = \dfrac{\partial\chi^k}{\partial X^l}\boldsymbol{i}_k \otimes \boldsymbol{i}^l$.

deformation these volumes again constitute a connected region. The natural step is to introduce another rotations, independent from $\boldsymbol{\chi}$, and to associate them with "microparticles".

Note, that rotation of infinitesimal fiber, represented by translation vector $d\mathbf{X}$, can be determined within Rodrigues' rotation formula[12] [33]:

$$d\mathbf{X}' = \cos\theta d\mathbf{X} + \sin\theta(\boldsymbol{k}\times d\mathbf{X}) + (\boldsymbol{k}\cdot d\mathbf{X})(1-\cos\theta)\boldsymbol{k}. \tag{4.4}$$

Here $\boldsymbol{k}$ is a unit vector directed along the rotation axis and θ is rotation angle. Thus, instead of dealing with second-rank tensor field $\boldsymbol{O}$ one can deal with pair $(\theta,\, \boldsymbol{k})$, which consists of rotation angle and axis direction. To this end, we introduce vector field $\boldsymbol{\varphi_O} := \theta\boldsymbol{k}$, which we would use as a basis for the future generalization.

Let us discuss relation between rotation tensor $\boldsymbol{O}$ and pair $(\theta,\, \boldsymbol{k})$ in more detail. The following formula allows to obtain $\boldsymbol{O}$ upon pair[13] $(\theta,\, \boldsymbol{k})$:

$$\boldsymbol{O} = \cos\theta\,\mathbf{1} + (1-\cos\theta)\boldsymbol{k}\otimes\boldsymbol{k} + \sin\theta\boldsymbol{K},$$

where $\boldsymbol{K}$ is second-rank tensor such that[14] $\boldsymbol{K}\boldsymbol{v} = \boldsymbol{k}\times\boldsymbol{v}$. Conversely, suppose that $\boldsymbol{O}$ is known. To determine pair $(\theta,\, \boldsymbol{k})$, one needs to do the following steps [33]:

(1) Solve eigenvalue problem $\boldsymbol{O}\boldsymbol{k} = \boldsymbol{k}$ under condition $\|\boldsymbol{k}\| = 1$. This gives $\boldsymbol{k}$.

(2) The angle θ is obtained from the solution of equation $\sin\theta = -\dfrac{\operatorname{tr}(\boldsymbol{K}\boldsymbol{O})}{2}$.

15°. Micropolar model. Within the micropolar theory of elasticity, in contrast with conventional (symmetric) theory, it is assumed that the orientation of the elementary volume is determined independently. One introduces independent vector field $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{X}, t)$ and considers it as an additional generalized coordinate. In this case the density of Lagrangian $\mathscr{L}$ is a function of material coordinates $\mathbf{X}$, time t, fields $\boldsymbol{\chi}(\mathbf{X}, t)$, $\boldsymbol{\varphi}(\mathbf{X}, t)$ and their first gradients, i. e.

$$\mathscr{L} = \mathscr{L}(\mathbf{X}, t, \boldsymbol{\chi}, \boldsymbol{\nabla}\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}). \tag{4.5}$$

Note, that vector field $\boldsymbol{\varphi_O}$, associated with orthogonal tensor $\boldsymbol{O}$ from (4.2) through Rodrigues' formula, is independent from $\boldsymbol{\varphi}$. The independent field $\boldsymbol{\varphi}$ can be regarded as a section of vector bundle (see paragraph **12**°).

16°. Generalized micropolar model. One can give the following geometric interpretation of the kinematics of the micropolar continuum. The non-deformable orthonormal trihedrons (triples of non-coplanar vectors) are associated with points of the medium, that are oriented along the base vectors in the reference state, and when the medium is deformed, change its spatial orientation. A more general interpretation may be given, when one allows these trihedrons to distort.

Suppose that one has fixed some orthonormal basis $(\boldsymbol{i}_k)_{k=1}^3$. Then for every trihedron $(\boldsymbol{d}_k)_{k=1}^3$ of non-coplanar vectors there exists invertible linear map $\boldsymbol{L} : \mathbb{V} \to \mathbb{V}$, such that $\boldsymbol{L}[\boldsymbol{i}_k] = \boldsymbol{d}_k$. In dyadic representation, $\boldsymbol{L} = \boldsymbol{d}_k \otimes \boldsymbol{i}^k$. Thus, instead of deformable trihedrons one can consider invertible linear maps, which are elements of general linear group. Meanwhile, the following issue arise here. The physical space $\mathbb{E}$ is too narrow to contain simultaneously places of particles and linear operators, associated with them. If we think of shapes, as of sets, which elements are pairs $(X,\, \boldsymbol{L})$, then these sets cannot be defined as

[12]One can obtain another version of formula (4.4). If put $\tau = \tan\frac{\theta}{2}$, then from (4.4) directly follows that

$$d\mathbf{X}' = \frac{1}{1+\tau^2}\left[(1-\tau^2)d\mathbf{X} + 2\tau(\boldsymbol{k}\times d\mathbf{X}) + 2\tau^2(\boldsymbol{k}\cdot d\mathbf{X})\boldsymbol{k}\right].$$

After some algebraic transformations one gets [32]

$$d\mathbf{X}' = d\mathbf{X} + \frac{1}{1+\frac{1}{4}\vartheta^2}\boldsymbol{\vartheta}\times\left(d\mathbf{X} + \frac{1}{2}\boldsymbol{\vartheta}\times d\mathbf{X}\right),$$

where $\boldsymbol{\vartheta} = 2\tau\boldsymbol{k}$, $\vartheta = \|\boldsymbol{\vartheta}\| = 2\tau$. The vector $\boldsymbol{\vartheta}$ is usually called as finite rotation vector, although its magnitude have no common with real rotation angle.

[13]If some orthonormal basis $(\boldsymbol{i}_s)_{s=1}^3$ in $\mathbb{V}$ has chosen, then, according to (4.4), operator $\boldsymbol{O}$ is represented by matrix

$$[\boldsymbol{O}] = \left(\begin{array}{c|c|c} (1-\cos\theta)k_1^2+\cos\theta & k_1k_2(1-\cos\theta)-k_3\sin\theta & k_1k_3(1-\cos\theta)+k_2\sin\theta \\ k_1k_2(1-\cos\theta)+k_3\sin\theta & (1-\cos\theta)k_2^2+\cos\theta & k_2k_3(1-\cos\theta)-k_1\sin\theta \\ k_1k_3(1-\cos\theta)-k_2\sin\theta & k_2k_3(1-\cos\theta)+k_1\sin\theta & (1-\cos\theta)k_3^2+\cos\theta \end{array}\right),$$

where k_1, k_2, k_3 are components of $\boldsymbol{k}$.

[14]In matrix form $\boldsymbol{K}$ is given by

$$[\boldsymbol{K}] = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}.$$

images of embeddings to physical space. Usually one employs the artificial construction, when "shape" is a region of physical space, while linear operators are considered separately, being associated with each point of this region. This reasoning is suitable for conventional problems, but hardly suitable to situations when one considers incompatible deformations. To use methodology of non-Euclidean shape one needs to introduce somehow shapes, that contain places and linear operators simultaneously.

To fix the issue it seems appropriate to consider enhanced physical space $L\mathbb{E}$, which is the total space of frame bundle $(L\mathbb{E}, \mathbb{E}, \pi_{\mathbb{E}}, \mathrm{GL}(3; \mathbb{R}), ., \lhd_{\mathbb{E}})$ discussed in paragraph **8°**. In this case configuration can be defined in usual manner as embedding $\kappa : \mathfrak{B} \to L\mathbb{E}$ and images of configurations are shapes of the body. This viewpoint is systematically developed in [34].

Any configuration $\kappa : \mathfrak{B} \to L\mathbb{E}$ defines an embedding $\varkappa : \mathfrak{B} \to \mathbb{E}$, conventional configuration, as follows: $\varkappa := \pi_{\mathbb{E}} \circ \kappa$. Thus, one indeed deals with enhanced picture, which contains all information about places of points. In the particular case, when $L\mathbb{E} = \mathbb{E} \times \mathrm{GL}(3; \mathbb{R})$, one has the following representation for κ: $\kappa(\mathfrak{X}) = (\varkappa(\mathfrak{X}); \boldsymbol{L}(\mathfrak{X}))$. Here $\varkappa : \mathfrak{B} \to \mathbb{E}$ is conventional configuration and $\boldsymbol{L} : \mathfrak{B} \to \mathrm{GL}(3; \mathbb{R})$ corresponds to field of trihedrons.

If one chooses natural coordinates on $L\mathbb{E}$, generated by coordinate map $\varphi_{L\mathbb{E}} : L\mathbb{E} \to \mathbb{R}^3 \times \mathbb{R}^9$, and some coordinates on the body manifold $\mathfrak{B}$, generated by coordinate map $\varphi_{\mathfrak{B}} : \mathfrak{B} \to \mathbb{R}^n$, then coordinate representation of configuration $\kappa : \mathfrak{B} \to L\mathbb{E}$ is composition

$$\widetilde{\kappa} = \varphi_{L\mathbb{E}} \circ \kappa \circ \varphi_{\mathfrak{B}}^{-1}, \quad \widetilde{\kappa}(\mathfrak{X}^1, ..., \mathfrak{X}^n) = (\widetilde{\varkappa}^i(\mathfrak{X}^1, ..., \mathfrak{X}^n); [L]^i{}_j(\mathfrak{X}^1, ..., \mathfrak{X}^n)),$$

where $i, j = 1, 2, 3$ and $\widetilde{\varkappa} : \mathbb{R}^n \to \mathbb{R}^3$, $[L] : \mathbb{R}^n \to \mathbb{R}^9$ are some mappings, uniquely defined by κ. Moreover, $\widetilde{\varkappa}$ is coordinate representation for $\varkappa$: $\widetilde{\varkappa} = \varphi_{\mathbb{E}} \circ \varkappa \circ \varphi_{\mathfrak{B}}^{-1}$, where $\varphi_{\mathbb{E}} : \mathbb{E} \to \mathbb{R}^3$ is coordinate map for physical space.

With a view to describe micropolar body with incompatible deformations in the framework of geometric continuum mechanics one needs to enhance the body manifold as well, since origin of incompatibility is more general here, as it involves particles and trihedrons, while body manifold contains information of particles only. Thus, one arrives at the frame bundle $(L\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, \mathrm{GL}(n; \mathbb{R}), ., \lhd_{\mathfrak{B}})$. In this case embeddings $L\varkappa : L\mathfrak{B} \to L\mathbb{E}$ represent enhanced configurations[15] (see paragraph **25°** of the Appendix). Within suitable coordinates, $L\varkappa$ is represented as mapping $\widetilde{L\varkappa} = \varphi_{L\mathbb{E}} \circ L\varkappa \circ \varphi_{L\mathfrak{B}}^{-1}$, where $\varphi_{L\mathfrak{B}} : L\mathfrak{B} \to \mathbb{R}^n \times \mathbb{R}^9$ is coordinate map on total space $L\mathfrak{B}$. All above reasonings are illustrated on Fig. 3. The dashed lines from $L\mathfrak{B}$ illustrate that this part

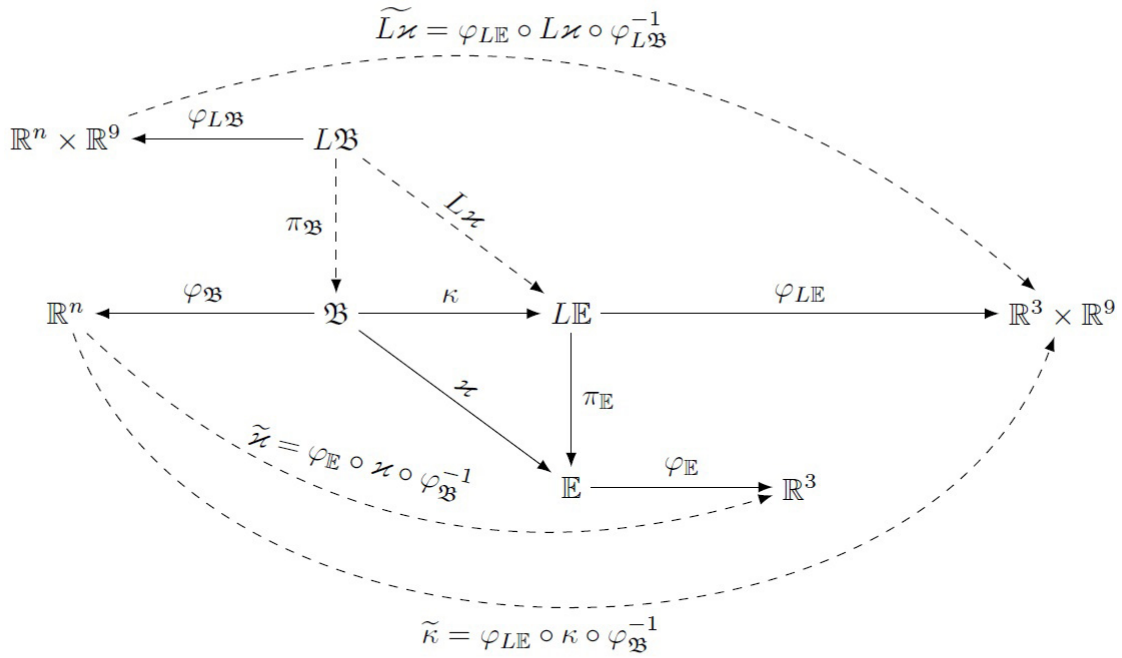

Fig. 3. Embedding of generalized micropolar body into enhanced physical space

of the figure corresponds to the case of incompatible deformations. One can obtain the conventional case of micropolar continuum if replaces general linear group $\mathrm{GL}(3; \mathbb{R})$ by special orthogonal group $\mathrm{SO}(3)$.

17°. Micromorphic model. In reasonings of paragraph **16°** we have associated with each particle some invertible linear transformation $\boldsymbol{L} : \mathbb{V} \to \mathbb{V}$, which is a counterpart for deformable trihedron of directors.

[15] It would be too restrictive to assume that embedding $L\varkappa$ is related with configuration $\kappa : \mathfrak{B} \to L\mathbb{E}$ as $L\varkappa = \kappa \circ \pi_{\mathfrak{B}}$. Indeed, in this case the image $L\varkappa(L\mathfrak{B})$ would be a surface in $L\mathbb{E}$, not an open set.

Within this formalism, a motion of such the solid is represented by pair $(\varkappa(\mathfrak{X}, t), \boldsymbol{L}(\mathfrak{X}, t))$, in which $\varkappa : \mathfrak{B}\times \times]t_1, t_2[\to \mathbb{E}$ is family of conventional configurations and $\boldsymbol{L} : \mathfrak{B}\times]t_1, t_2[\to \mathrm{GL}(3; \mathbb{R})$ is family of linear maps. Meanwhile, one can adopt another point of view. First, one can choose reference directors from some m-dimensional Euclidean vector space $\mathbb{F}$ instead of choosing them from translation vector space $\mathbb{V}$. In actual picture one still observes trihedrons of vectors from $\mathbb{V}$. Second, one can suppose that relation between abstract directors, i. e., vectors from $\mathbb{F}$, and real directors, i. e., vectors from $\mathbb{V}$, is linear. The corresponding linear transformation is defined by a second-rank tensor $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}(\mathbf{X}, t) \in \mathrm{Lin}(\mathbb{F}; \mathbb{V})$, operating in a m-dimensional Euclidean vector space $\mathbb{F}$. This tensor is referred to as the tensor of microdeformations[16].

In view of the above, the density of Lagrangian can be described as:

$$\mathscr{L} = \mathscr{L}(\mathbf{X}, t, \boldsymbol{\chi}, \boldsymbol{\nabla}\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\nabla}\boldsymbol{\mathcal{X}}, \dot{\boldsymbol{\mathcal{X}}}). \tag{4.6}$$

It seems appropriate to mention about relation between field $\boldsymbol{\mathcal{X}} : \mathbb{S}_R \to \mathrm{Lin}(\mathbb{F}; \mathbb{V})$, the argument of Lagrangian density (4.6), and field $\boldsymbol{\mathcal{M}} : \mathbb{S}_R \to \mathrm{Lin}(\mathbb{F}; \mathbb{F})$ from (3.5). Values of both fields are linear maps that transform reference orientations to actual orientations. It is assumed here that orientations are vectors from some abstract vector space $\mathbb{F}$. Meanwhile, in conventional theories of elasticity orientations are considered to be visible deformable trihedrons of translation vectors, i. e., elements from $\mathbb{V}$. By this reason, we suppose the following relation between fields $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{M}}$: $\boldsymbol{\mathcal{X}} = \mathrm{In} \circ \boldsymbol{\mathcal{M}}$. Here $\mathrm{In} : \mathbb{S}_R \to \mathrm{Lin}(\mathbb{F}; \mathbb{V})$ is field of linear mappings, that perform inclusion of abstract orientations from $\mathbb{F}$ into visible elements from $\mathbb{V}$. The form of this field depends on physical nature of the model.

There is another way of justifying microdeformations, when one considers particles as aggregates, that consist of microparticles [35]. The body manifold $\mathfrak{B}$ contains labels of particles and its structure is too narrow to fit microparticles also. By this reason, suppose that each particle can be formalized as 3-dimensional Euclidean vector space $\mathbb{F}$, which, in turn, contains labels of microparticles, and introduce material vector bundle $(V\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, \mathbb{F})$ over $\mathfrak{B}$. Here each fiber $V\mathfrak{B}_{\mathfrak{X}}$ over $\mathfrak{X} \in \mathfrak{B}$ represents single cell, associated with label $\mathfrak{X}$. The distinction between particles and microparticles requires to introduce two types of configurations. The first type is conventional configuration represented by embedding $\varkappa : \mathfrak{B} \to \mathbb{E}$, $\mathfrak{X} \mapsto \varkappa(\mathfrak{X})$, associated with particles. The second type, for $\mathfrak{X} \in \mathfrak{B}$, is an embedding $\varkappa^{\mathbb{F}}_{\mathfrak{X}} : \mathbb{F} \to \mathbb{V}$, $\boldsymbol{U} \mapsto \varkappa^{\mathbb{F}}_{\mathfrak{X}}(\boldsymbol{U})$. We suppose the following calibration condition: $\varkappa^{\mathbb{F}}_{\mathfrak{X}}(\mathbf{0}) = \mathbf{0}$. Then embedding $\varkappa^{\mathbb{F}}_{\mathfrak{X}}$ can be interpreted as assignment, that returns radius-vectors of microparticles within center of mass of the single cell $\mathfrak{X}$.

Note, that one cannot think of $\varkappa^{\mathbb{F}}_{\mathfrak{X}}$ as of embedding into physical manifold $\mathbb{E}$, since in this case there would not be clear distinction between particles and microparticles. Instead of this, one can treat microparticles as infinitesimal translation vectors emanating from center of mass of the cell $\varkappa(\mathfrak{X})$.

As it was mentioned, one cannot consider positions of particles and microparticles in physical space $\mathbb{E}$ simultaneously. Meanwhile, using vector bundles again, one can introduce vector bundle $V\mathbb{E}$ over $\mathbb{E}$ with model fiber $\mathbb{V}$. Then place and “microplace” can be unified into a single point from $V\mathbb{E}$. Both mapping $\varkappa$ and the family $(\varkappa^{\mathbb{F}}_{\mathfrak{X}})_{\mathfrak{X}\in\mathfrak{B}}$ of mappings induce global map $V\varkappa : V\mathfrak{B} \to V\mathbb{E}$, i. e., enhanced configuration, from manifold $V\mathfrak{B}$ to manifold $V\mathbb{E}$. In what follows we suppose that $V\mathfrak{B} = \mathfrak{B} \times \mathbb{F}$ and $V\mathbb{E} = \mathbb{E} \times \mathbb{V}$, i. e., vector bundles over $\mathfrak{B}$ and $\mathbb{E}$ are trivial.

To get conventional picture, described by Mindlin [35], one can provide the following modeling. As enhanced configuration is completely defined by mappings $\varkappa$ and $\varkappa^{\mathbb{F}}_{\mathfrak{X}}$, we can deal only with them. Introduce the mapping $\overline{\varkappa} : \mathfrak{B} \times \mathbb{F} \to \mathbb{E}$ by the relation

$$\overline{\varkappa}(\mathfrak{X}, \boldsymbol{U}) := \varkappa(\mathfrak{X}) + \varkappa^{\mathbb{F}}_{\mathfrak{X}}(\boldsymbol{U}).$$

This mapping returns fictive position of microparticle, equal to the sum of position of the corresponding cell and relative position of microparticle within the cell. Consider first order Taylor expansion of $\varkappa^{\mathbb{F}}_{\mathfrak{X}}(\boldsymbol{U})$: $\varkappa^{\mathbb{F}}_{\mathfrak{X}}(\boldsymbol{U}) = \varkappa^{\mathbb{F}}_{\mathfrak{X}}(\mathbf{0}) + \boldsymbol{\mathcal{X}}_{\mathfrak{X}}(\boldsymbol{U}) + \boldsymbol{o}(\|\boldsymbol{U}\|)$. Here $\boldsymbol{\mathcal{X}}_{\mathfrak{X}} : \mathbb{F} \to \mathbb{V}$ is linear map. Taking into account the calibration condition, one gets

$$\overline{\varkappa}(\mathfrak{X}, \boldsymbol{U}) := \varkappa(\mathfrak{X}) + \boldsymbol{\mathcal{X}}_{\mathfrak{X}}(\boldsymbol{U}) + \boldsymbol{o}(\|\boldsymbol{U}\|).$$

Thus, with up to $\boldsymbol{o}(\|\boldsymbol{U}\|)$, the fictive position of microparticle is defined by pair $(\varkappa, \boldsymbol{\mathcal{X}})$, where $\varkappa : \mathfrak{B} \to \mathbb{E}$ is conventional configuration and $\boldsymbol{\mathcal{X}} : \mathfrak{B} \times \mathbb{F} \to \mathbb{V}$ is tensor field of microdeformations. Therefore, the density of Lagrangian has the form (we identify body manifold with some of its shapes):

$$\mathscr{L} = \mathscr{L}(\mathbf{X}, \boldsymbol{U}, t, \boldsymbol{\chi}, \boldsymbol{\nabla}\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\nabla}\boldsymbol{\mathcal{X}}, \dot{\boldsymbol{\mathcal{X}}}).$$

[16]Let us make more clear the distinction between models discussed in paragraphs **16°** and **17°**. Linear maps in them play different roles:

- In **16°** linear map is a counterpart for directors triad;
- In **17°** linear map relates family $(\boldsymbol{f}_I)_{I=1}^m$ of abstract reference directors with family $(\boldsymbol{d}_i)_{i=1}^3$ of their spatial counterpart.

Believing that Lagrangian density is independent of $\boldsymbol{U}$, *i. e.*, cells are homogeneous and isotropic, we arrive at (4.6). If the cell, above that, is absolutely rigid, the equations are reduced to the equations of the micropolar continuum (4.5).

Reasonings, provided so far, are illustrated on Fig. 4. On the figure: $\varphi_{\mathbb{F}}$, $\varphi_{V\mathfrak{B}}$, $\varphi_{\mathfrak{B}}$, $\varphi_{\mathbb{V}}$ and $\varphi_{\mathbb{E}}$ are

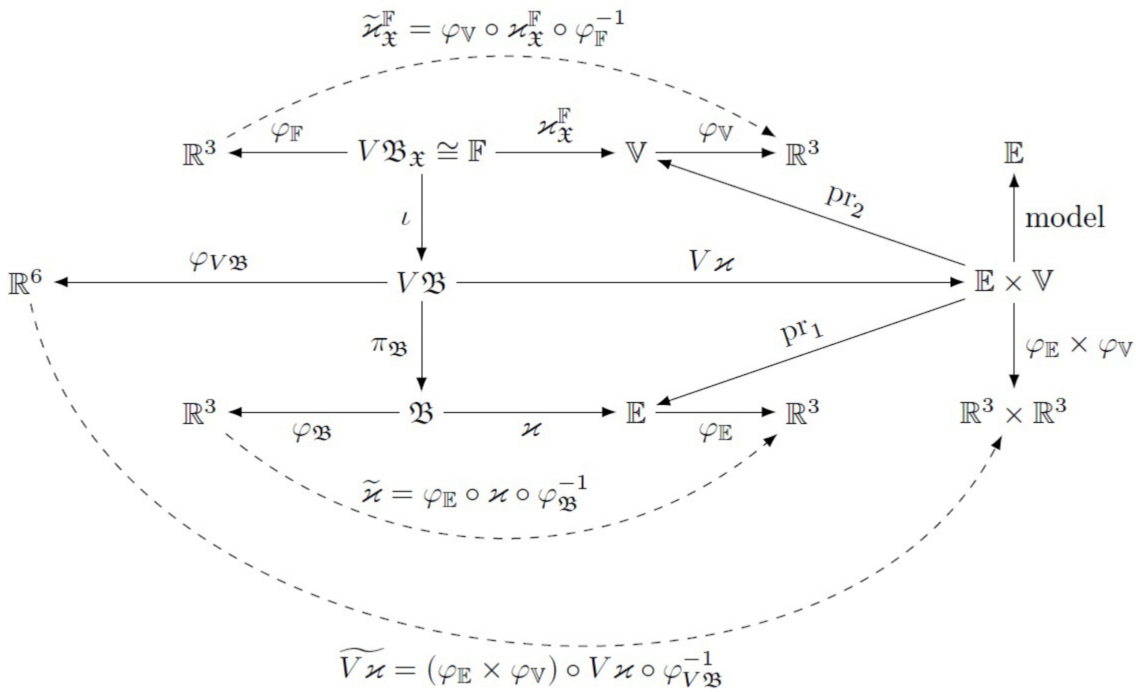

Fig. 4. Embedding of micromorphic body into physical space

coordinate mappings of manifolds indicated in lower indices. Mappings $\widetilde{\varkappa}^{\mathbb{F}}_{\mathfrak{X}}$, $\widetilde{\varkappa}$ and $\widetilde{V\varkappa}$ are coordinate representations of the corresponding mappings $\varkappa^{\mathbb{F}}_{\mathfrak{X}}$, $\varkappa$ and $V\varkappa$. The mapping ι is inclusion map. The word "model" on the right upper part of the figure indicates the transition from enhanced configuration $V\varkappa$ to fictive configuration $\overline{\varkappa}$.

18°. Shell models. Although there are many distinct models of shells, the fiber bundle formalism allows to think of these models as one. To this end a shell can be formalized as material fiber bundle $(F\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, \mathbb{F})$ where total space $F\mathfrak{B}$ is smooth manifold of dimension 3 and base $\mathfrak{B}$ has dimension 2. The manifold $F\mathfrak{B}$ contains all labels of shell particles, while manifold $\mathfrak{B}$ contains labels of points from reduction surface. Every fiber $F\mathfrak{B}_{\mathfrak{X}}$, one-dimensional manifold, consists of material transversal elements of the shell over point $\mathfrak{X}$.

In physical space $\mathbb{E}$ one observes shape $\widehat{\mathbb{S}}$ of the shell, the image of embedding $\kappa : F\mathfrak{B} \to \mathbb{E}$, and shape $\mathbb{S}$ of the body $\mathfrak{B}$, the image of embedding $\varkappa : \mathfrak{B} \to \mathbb{E}$. In general, κ and $\varkappa$ are independent and shape $\mathbb{S}$ may not be contained in $\widehat{\mathbb{S}}$. We refer to the shape $\mathbb{S}$ as reduction surface. Let $\varphi_{F\mathfrak{B}} : F\mathfrak{B} \to \mathbb{R}^3$ be coordinate map, that returns natural coordinates of points from $F\mathfrak{B}$. That is, for $p \in F\mathfrak{B}$ one has triple $(\mathfrak{X}^1,\, \mathfrak{X}^2,\, \xi) \in \mathbb{R}^3$, where $(\mathfrak{X}^1,\, \mathfrak{X}^2)$ are coordinates on the manifold $\mathfrak{B}$ and ξ is "transversal" coordinate, i. e., coordinate on model fiber $\mathbb{F}$. Let $\widehat{\kappa} : F\mathfrak{B} \to \widehat{\mathbb{S}}$ be mapping, that obtained from κ by restriction of codomain (see **9**°). Then define coordinate map $\varphi_{\widehat{\mathbb{S}}} : \widehat{\mathbb{S}} \to \mathbb{R}^3$ as $\varphi_{\widehat{\mathbb{S}}} = \varphi_{F\mathfrak{B}} \circ \widehat{\kappa}^{-1}$. Thus, points from shape $\widehat{\mathbb{S}}$ are represented by tuples $(\mathfrak{X}^1,\, \mathfrak{X}^2,\, \xi)$.

One arrives at the following general vectorial relation between positions of points from $\widehat{\mathbb{S}}$ and natural coordinates on $F\mathfrak{B}$:

$$\boldsymbol{r} = \boldsymbol{f}(\mathfrak{X}^1,\, \mathfrak{X}^2;\, \xi), \tag{4.7}$$

where $\boldsymbol{f}$ depends on the model chosen for description of configuration. The semicolon separates coordinates on reduction surface from transversal coordinate. If one assumes that transversal elements are associated with normal elements, then one gets the following approximation for configuration κ: $\boldsymbol{r} = \boldsymbol{\rho}(\mathfrak{X}^1,\, \mathfrak{X}^2) + \xi\, \boldsymbol{n}(\mathfrak{X}^1,\, \mathfrak{X}^2)$, where $\boldsymbol{\rho}(\mathfrak{X}^1,\, \mathfrak{X}^2)$ is position vector of reduction surface and $\boldsymbol{n}$ is unit normal vector field.

In general, the choice of particular approximation for (4.7) is provided in two stages. One pulls out the reduction surface $\mathbb{S}$ from $\widehat{\mathbb{S}}$ and considers it as base manifold for some vector bundle $(V\mathbb{S},\, \mathbb{S},\, \pi_{\mathbb{S}},\, \mathbb{F}_1)$ with model fiber $\mathbb{F}_1$, an m-dimensional vector space. Elements of the space $\mathbb{F}_1$ serve as objects, which are used to approximate partial field $\boldsymbol{f}(\mathfrak{X}^1,\, \mathfrak{X}^2;\, \cdot) : \xi \mapsto \boldsymbol{f}(\mathfrak{X}^1,\, \mathfrak{X}^2;\, \xi)$. In particular, one may consider some functional Hilbert space. Choosing some basis, e.g., Legendre polynomials, one then restricts its attention in linear span of dimension m. This is what exactly $\mathbb{F}_1$ is. In the next step one combines fields $\boldsymbol{f}(\cdot;\,\xi) : (\mathfrak{X}^1,\,\mathfrak{X}^2) \mapsto \boldsymbol{f}(\mathfrak{X}^1,\,\mathfrak{X}^2;\,\xi)$ and $\boldsymbol{f}(\mathfrak{X}^1,\,\mathfrak{X}^2;\,\cdot) : \xi \mapsto \boldsymbol{f}(\mathfrak{X}^1,\,\mathfrak{X}^2;\,\xi)$ into one global field from $\mathbb{R}^3$ to $\mathbb{E}$. The image of this field is denoted by $\overline{\mathbb{S}}$. The obtained field is the result of approximation. It is expected that after modeling $\widehat{\mathbb{S}} = \overline{\mathbb{S}}$, but it is worth to note, that the shape $\overline{\mathbb{S}}$ is not the image of embedding from $F\mathfrak{B}$.

Suppose that $(e_k)_{k=1}^{\infty}$ is some Schauder basis of Hilbert space $L^2[-1,\,1]$ (not necessary orthonormal). Choosing $m \in \mathbb{N}$, we replace $\boldsymbol{f}$ from (4.7) by additive decomposition $\boldsymbol{r} = \boldsymbol{\rho}(\mathfrak{X}^1,\,\mathfrak{X}^2) + \boldsymbol{\rho}_1(\mathfrak{X}^1,\,\mathfrak{X}^2;\,\xi)$, where $\boldsymbol{\rho}_1(\mathfrak{X}^1,\,\mathfrak{X}^2;\,\xi) = \sum_{k=1}^{m} \boldsymbol{c}_k(\mathfrak{X}^1,\,\mathfrak{X}^2) e_k(\xi)$. Thus,

$$\boldsymbol{r} = \boldsymbol{\rho}(\mathfrak{X}^1,\,\mathfrak{X}^2) + \sum_{k=1}^{m} \boldsymbol{c}_k(\mathfrak{X}^1,\,\mathfrak{X}^2) e_k(\xi).$$

Here $\boldsymbol{c}_k(\mathfrak{X}^1,\,\mathfrak{X}^2)$ are vectorial coefficients of the expansion, which components define elements from vector space $\mathbb{F}_1$.

All reasonings provided above are illustrated on Fig. 5. On the figure, symbols φ with lower indices

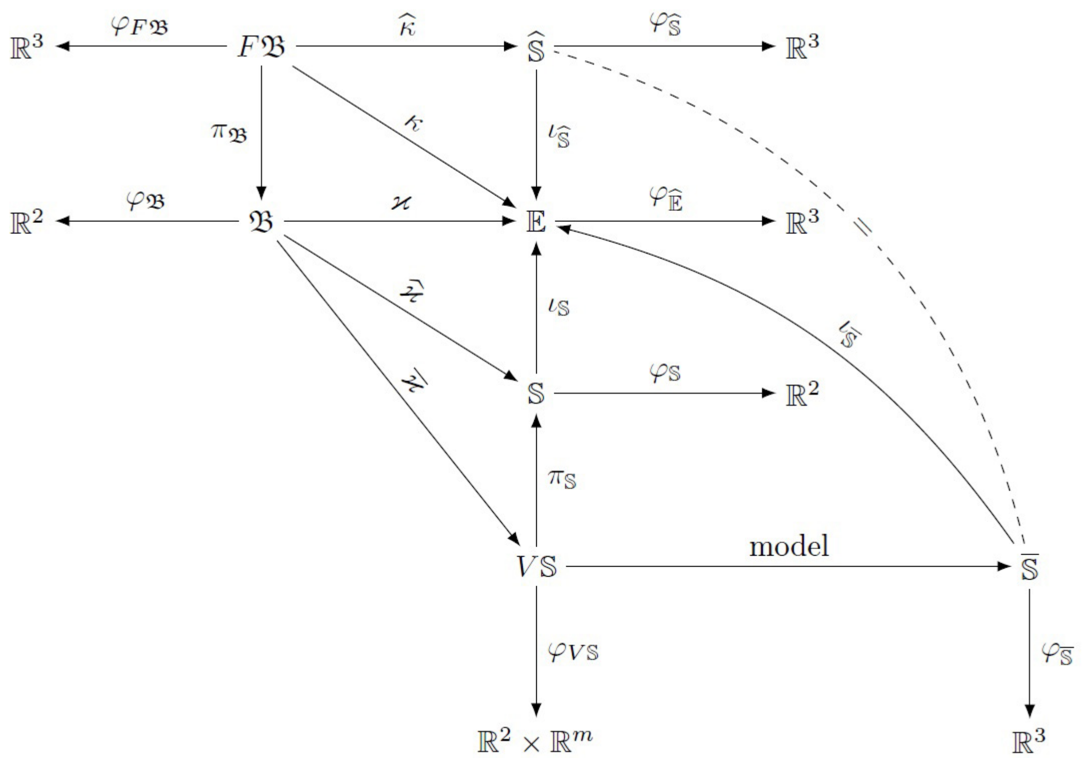

Fig. 5. Shell modeling scheme

denote the corresponding coordinate maps. Mappings ι are inclusion maps and "model" denotes the process of transition from manifold $V\mathbb{S}$ to the manifold $\overline{\mathbb{S}}$.

5. Incompatible Deformations

5.1. Local Discharging

19°. Family of shapes. In conventional elasticity one assumes that body $\mathfrak{B}$ has stress-free shape $\mathbb{S}_R$ in Euclidean physical space $\mathbb{E}$. Like all other shapes, this shape is considered as a subspace of physical manifold, with metric and connection induced from it:

$$\mathbb{S}_R = (S_R,\, \mathcal{T}_E|_{S_R},\, \mathcal{D}_E|_{S_R},\, \boldsymbol{g}_E|_{S_R},\, \nabla_E|_{S_R}).$$

Here S_R is the underlying set of the shape, while other elements of the tuple are, respectively, topology, smooth structure, metric and connection of Euclidean space $\mathbb{E}$ induced to S_R. In this case deformation is a mapping

$$\gamma : (S_R,\, \mathcal{T}_E|_{S_R},\, \mathcal{D}_E|_{S_R},\, \boldsymbol{g}_E|_{S_R},\, \nabla_E|_{S_R}) \to (S,\, \mathcal{T}_E|_{S},\, \mathcal{D}_E|_{S},\, \boldsymbol{g}_E|_{S},\, \nabla_E|_{S}),$$

between two open (hereafter we consider the case $\dim \mathfrak{B} = 3$) submanifolds of $\mathbb{E}$ with Euclidean geometry.

If solid has defects, e.g., dislocations, disclinations, etc., then the assumption of existence of *global* Euclidean stress-free shape fails. Before stating weaker assumption, let us provide auxiliary consideration.

Suppose that one have chosen some shape $\mathbb{S}_I$ in Euclidean physical space $\mathbb{E}$, the image of configuration $\varkappa_I : \mathfrak{B} \to \mathbb{E}$. We refer to it as *intermediate shape*. Moreover, let $\mathbb{S}_R$ and $\mathbb{S}$ be, respectively, reference and actual shapes in $\mathbb{E}$, which are images of configurations $\varkappa_R, \varkappa : \mathfrak{B} \to \mathbb{E}$ respectively. The shape $\mathbb{S}_R$ is assumed to be stress-free, while $\mathbb{S}_I$ and $\mathbb{S}$ are self-stressed in general. Define maps

$$\chi = \varkappa \circ \widehat{\varkappa}_I^{-1} : \mathbb{S}_I \to \mathbb{S}, \quad \xi = \varkappa_R \circ \widehat{\varkappa}_I^{-1} : \mathbb{S}_I \to \mathbb{S}_R,$$

which are deformations from the intermediate shape to actual and reference shapes respectively. Then ξ corresponds to relaxation from the shape $\mathbb{S}_I$ to the shape $\mathbb{S}_R$, and the composition $\gamma = \chi \circ \xi^{-1} : \mathbb{S}_R \to \mathbb{S}$ is the conventional deformation from stress-free reference shape to the actual shape.

Instead of considering motion $\gamma : \mathbb{S}_R \times [t_0, t_1] \to \mathbb{E}$ as a field variable, one can choose mappings $\chi : \mathbb{S}_I \times$ $\times [t_0, t_1] \to \mathbb{E}$ and $\xi : \mathbb{S}_I \times [t_0, t_1] \to \mathbb{E}$, defined on $\mathbb{S}_I$, to be field variables. Thus, the Lagrangian density in (4.1) is replaced by the following function:

$$\mathscr{L} = \mathscr{L}(X, t, \chi, \boldsymbol{\nabla}\chi, \dot{\chi}, \xi, \boldsymbol{\nabla}\xi, \dot{\xi}). \tag{5.1}$$

Assuming independence on rigid translations, one can replace (5.1) by

$$\mathscr{L} = \mathscr{L}(X, t, \boldsymbol{\nabla}\chi, \dot{\chi}, \boldsymbol{\nabla}\xi, \dot{\xi}). \tag{5.2}$$

Independence on rigid rotations, in turn, allows to replace (5.2) by

$$\mathscr{L} = \mathscr{L}(X, t, \boldsymbol{B}, \dot{\chi}, \boldsymbol{\beta}, \dot{\xi}). \tag{5.3}$$

In (5.3) $\boldsymbol{B}$ and $\boldsymbol{\beta}$ are, respectively, $\boldsymbol{B} = \boldsymbol{\nabla}\chi\boldsymbol{\nabla}\chi^{\mathrm{T}}$ and $\boldsymbol{\beta} = \boldsymbol{\nabla}\xi\boldsymbol{\nabla}\xi^{\mathrm{T}}$.

Now, bearing in mind the auxiliary consideration, consider the case, when all Euclidean shapes of the body are self-stressed. We suppose the weaker assumption, to which we refer as *principle of local discharging*: there exist a shape $\mathbb{S}_I$ in Euclidean space and a family $(\gamma^{(X)})_{X\in\mathbb{S}_I}$ of deformations $\gamma^{(X)} : \mathbb{S}_I \to \mathbb{S}^{(X)}$ from the shape $\mathbb{S}_I$ to some shapes $\mathbb{S}^{(X)}$, such that the infinitesimal neighborhood of point $\gamma^{(X)}(Y)|_{Y=X}$ is stress-free for any $X \in \mathbb{S}_I$.

For each deformation $\gamma^{(X)}$ one can obtain deformation gradient $\boldsymbol{F}^{(X)} = \boldsymbol{\nabla}\gamma : \mathbb{S}_I \to \mathrm{Lin}(\mathbb{V}; \mathbb{V})$, $\boldsymbol{F}^{(X)} : Y \mapsto \boldsymbol{F}^{(X)}_Y$. Due to the principle of local discharging, tensor field $\boldsymbol{F}^{(X)}$ satisfies the following property: the linear map $\boldsymbol{F}^{(X)}_Y|_{Y=X} : \mathbb{V} \to \mathbb{V}$ transforms infinitesimal neighborhood of point $X \in \mathbb{S}_I$ to infinitesimal neighborhood of point $\gamma^{(X)}(Y)|_{Y=X}$, which is stress-free. Synthesize upon the family $(\boldsymbol{F}^{(X)})_{X\in\mathbb{S}_I}$ the new field $\boldsymbol{H}$ of linear mappings as follows:

$$\boldsymbol{H} : \mathbb{S}_I \to \mathrm{Lin}(\mathbb{V}; \mathbb{V}), \quad X \mapsto \boldsymbol{H}_X := \boldsymbol{F}^{(X)}_Y|_{Y=X}. \tag{5.4}$$

According to definition, for each $X \in \mathbb{S}_I$ the linear map $\boldsymbol{H}_X$ transforms infinitesimal neighborhood of X to stress-free state. We refer to the field $\boldsymbol{H}$ as field of local deformations. Since it is synthesized upon distinct deformation gradients, it is not the gradient of some map $\xi : \mathbb{S}_I \to \mathbb{E}$ that globally relaxes shape $\mathbb{S}_I$ to stress-free state.

Within our considerations, the Lagrangian density (5.1) of conventional elasticity should be modified to the form

$$\mathscr{L} = \mathscr{L}(X, t, \gamma, \boldsymbol{\nabla}\gamma, \dot{\gamma}, \boldsymbol{H}, \dot{\boldsymbol{H}}).$$

Here $\gamma : \mathbb{S}_I \times [t_0, t_1] \to \mathbb{E}$ and $\boldsymbol{H} : \mathbb{S}_I \times [t_0, t_1] \to \mathrm{Lin}(\mathbb{V}; \mathbb{V})$ are, respectively, the motion of the shape $\mathbb{S}_I$ and time-dependent field of local deformations. Both are considered as independent field quantities.

20°. Principle of local discharging: micromorphic kinematics. For definiteness, suppose that we deal with enhanced physical space and shapes which are modeled as vector bundles. Then enhanced deformation between two shapes is represented by formula (5.10) from Appendix, while micromorphic deformation is represented by formula (3.5):

$$V\gamma(X) = (\gamma(X), \boldsymbol{\mathcal{M}}(X)[U_X]),$$

where γ is conventional deformation and $\boldsymbol{\mathcal{M}}(X) \in \mathrm{Lin}(\mathbb{F}; \mathbb{F})$ is linear map, which relates reference and actual directors. In this case the principle of local discharging is assumed to be of the form: there exist a shape $\mathbb{S}_I$ in physical space $\mathbb{E}$, a family $(\gamma^{(X)})_{X\in\mathbb{S}_I}$ of deformations $\gamma^{(X)} : \mathbb{S}_I \to \mathbb{S}^{(X)}$ from the shape $\mathbb{S}_I \subset \mathbb{E}$ to some shapes $\mathbb{S}^{(X)}$ and a family $(\boldsymbol{\mathcal{M}}^{(X)})_{X\in\mathbb{S}_I}$ of directors $\boldsymbol{\mathcal{M}}^{(X)} : \mathbb{S}_I \to \mathrm{Lin}(\mathbb{F}; \mathbb{F})$, such that the infinitesimal neighborhood of point $V\gamma^{(X)}(Y)|_{Y=X} = (\gamma^{(X)}(Y)|_{Y=X}, \boldsymbol{\mathcal{M}}^{(X)}(Y)|_{Y=X})$ is stress-free for any $X \in \mathbb{S}_I$.

We utilize the procedure of synthesizing to deformation gradients $\boldsymbol{F}^{(X)} = \boldsymbol{\nabla}\gamma : \mathbb{S}_I \to \mathrm{Lin}(\mathbb{V}; \mathbb{V})$, and to director gradients $\boldsymbol{\nabla}\boldsymbol{\mathcal{M}}^{(X)} : \mathbb{S}_I \to \mathrm{Lin}(\mathbb{V}; \mathrm{Lin}(\mathbb{F}; \mathbb{F}))$. This results in field (5.4) of local deformations (like in the case of simple solid), and in field

$$\boldsymbol{D} : \mathbb{S}_I \to \mathrm{Lin}(\mathbb{V}; \mathrm{Lin}(\mathbb{F}; \mathbb{F})), \quad X \mapsto \boldsymbol{D}_X = \boldsymbol{\nabla}_Y\boldsymbol{\mathcal{M}}^{(X)}|_{Y=X},$$

synthesized upon gradients of microdeformation tensors.

In the case of micromorphic kinematics the Lagrangian density (4.6) of micromorphic elasticity is modified to the form

$$\mathscr{L} = \mathscr{L}(X,\, t,\, \gamma,\, \boldsymbol{\nabla}\gamma,\, \dot{\gamma},\, \boldsymbol{\mathcal{M}},\, \boldsymbol{\nabla}\boldsymbol{\mathcal{M}},\, \dot{\boldsymbol{\mathcal{M}}},\, \boldsymbol{H},\, \dot{\boldsymbol{H}},\, \boldsymbol{D},\, \boldsymbol{\nabla}\boldsymbol{D},\, \dot{\boldsymbol{D}}).$$

Here $\gamma : \mathbb{S}_I \times [t_0,\, t_1] \to \mathbb{E}$, $\boldsymbol{\mathcal{M}} : \mathbb{S}_I \times [t_0,\, t_1] \to \mathrm{Lin}(\mathbb{F};\, \mathbb{F})$ and $\boldsymbol{H} : \mathbb{S}_I \times [t_0,\, t_1] \to \mathrm{Lin}(\mathbb{V};\, \mathbb{V})$, $\boldsymbol{D} : \mathbb{S}_I \times [t_0,\, t_1] \to \mathrm{Lin}(\mathbb{V};\, \mathrm{Lin}(\mathbb{F};\, \mathbb{F}))$ are, respectively, the motion of the shape $\mathbb{S}_I$ and time-dependent fields of local deformations. Maps γ, $\boldsymbol{\mathcal{M}}$, $\boldsymbol{H}$, and $\boldsymbol{D}$ are considered as independent field quantities.

5.2. Geometrical Viewpoint

21°. Simple body. Considerations of section 5.1. were based on the rejection of using the global stress-free shape. Instead of this, one deals with family of locally stress-free shapes. Meanwhile, the concept of global stress-free shape is attractive enough, since one can deal with material description of processes, which is commonly convenient in non-linear elasticity. To allow global stress-free shapes, one just needs to go beyond Euclidean structure and consider manifolds equipped with non-Euclidean geometry. Choosing appropriate geometry, one obtains global stress-free shape and then deformations become embeddings of non-Euclidean stress-free shape to Euclidean space.

The stress-free shape is modeled on the intermediate shape $\mathbb{S}_I = (S_I,\, \mathcal{T}_E|_{S_I},\, \mathcal{D}_E|_{S_I},\, \boldsymbol{g}_E|_{S_I},\, \nabla_E|_{S_I})$. To do this, one needs to erase Euclidean structure from it. Thus results in manifold $M_R = (S_I,\, \mathcal{T}_E|_{S_I},\, \mathcal{D}_E|_{S_I})$, pure from any geometry. Let $\boldsymbol{H} : M_I \to \mathrm{Lin}(\mathbb{V};\, \mathbb{V})$ be field of local deformations. One can use it to introduce geometry on M_I.

Riemannian metric $\mathsf{G} : M_I \to T^*M_I \otimes T^*M_I$ is defined on M_I upon field of local deformations as follows:

$$\mathsf{G}_X(\mathsf{u},\, \mathsf{v}) := \boldsymbol{H}_X[\mathsf{u}] \cdot \boldsymbol{H}_X[\mathsf{u}], \tag{5.5}$$

for all $X \in M_I$ and $\mathsf{u}, \mathsf{v} \in T_X M_I$. Let us explain the action of “Euclidean” tensor $\boldsymbol{H}_X \in \mathrm{Lin}(\mathbb{V};\mathbb{V})$ on abstract tangent vector from $T_X M_I$. Since we consider the case $\dim \mathfrak{B} = 3$, the shape M_I is open submanifold of $\mathbb{E}$. Then there is natural vector space isomorphism $T_X M_I \cong \mathbb{V}$ [25]. By this reason, we tacitly identify element of $\mathrm{Lin}(\mathbb{V};\, \mathbb{V})$ with element of $\mathrm{Lin}(T_X M_I;\, \mathbb{V})$.

The value G_X of field G is non-degenerate symmetric positive-definite bilinear form on $T_X M_I$, that defines metric structure on $T_X M_I$. Upon tangent vectors $\mathsf{u},\, \mathsf{v} \in T_X M_I$, which are infinitesimal material fibers, it returns the inner product $\boldsymbol{u}\cdot\boldsymbol{v}$ of their relaxed images $\boldsymbol{u} = \boldsymbol{H}_X[\mathsf{u}]$ and $\boldsymbol{v} = \boldsymbol{H}_X[\mathsf{v}]$. Thus, one obtains lengths of material fibers and angles between them as if they where in relaxed state.

There are several possibilities to introduce affine connection ∇ on M_I [26, 36]. One of them, the Weitzenböck connection, is chosen in the study. Its connection functions in coordinate frame are given by relations

$$\Gamma^i{}_{jk} = -\boldsymbol{H}^c{}_k \partial_{X^j} [\boldsymbol{H}^{-1}]^i{}_c = [\boldsymbol{H}^{-1}]^i{}_c \partial_{X^j} \boldsymbol{H}^c{}_k, \tag{5.6}$$

where $\boldsymbol{H}^{-1} : X \mapsto \boldsymbol{H}_X^{-1}$ is field of inverse local deformations. The deviation of this connection from Euclidean one is completely characterized by torsion tensor field $\mathfrak{T}$. Thus, $\mathfrak{T}$ may serve as the measure of inhomogeneity.

Thus, one arrives at non-Euclidean space $\mathcal{S} = (M_I,\, \mathsf{G},\, \nabla)$, to which we refer as *non-Euclidean reference shape*. Any deformation is then an embedding $\lambda : \mathcal{S} \to \mathbb{E}$ of non-Euclidean manifold into Euclidean one. It can be decomposed into two deformations,

$$\lambda = \gamma \circ \psi,$$

where $\psi : \mathcal{S} \to \mathbb{S}_I$ is a smooth map that “imprints” non-Euclidean shape into Euclidean space. More formally, it replaces the structure $(M_I,\, \mathsf{G},\, \nabla)$ with underlying manifold M_I by the structure[17] $(M_I,\, \boldsymbol{g}_E|_{S_I},\, \nabla_E|_{S_I})$ with the same underlying manifold M_I, such that $X \mapsto X$. Other map $\gamma : \mathbb{S}_I \to \mathbb{E}$ is conventional deformation between Euclidean shapes.

Since $\mathcal{S}$ is manifold with non-Euclidean parallel transport rule, one cannot use Euclidean differentiation operator $\boldsymbol{\nabla}$. Instead of it, one can use tangent map operation T [25]. Being applied to deformation λ, it results in the map $T\lambda : T\mathcal{S} \to T\mathbb{E}$, which can be considered as the collection of linear maps $T_X\lambda : T_X\mathcal{S} \to T_{\lambda(X)}\mathbb{E}$, $X \in \mathcal{S}$, with matrices $[T\lambda]$ in coordinate frame defined as follows:

$$[T\lambda] = D(\varphi \circ \lambda \circ \sigma^{-1}),$$

where $\sigma : \mathcal{S} \to \mathbb{R}^3$ is coordinate map on $\mathcal{S}$ and $\varphi : \mathbb{E} \to \mathbb{R}^3$ is coordinate map on $\mathbb{E}$. The symbol D stands for total derivative in $\mathbb{R}^3$.

Assuming all considered fields as time-dependent, one can introduce the following Lagrangian density:

$$\mathscr{L} = \mathscr{L}(X,\, t,\, \lambda,\, T\lambda,\, \dot{\lambda},\, \mathsf{G},\, \dot{\mathsf{G}},\, \mathfrak{T},\, \dot{\mathfrak{T}}). \tag{5.7}$$

[17] Here $\boldsymbol{g}_E$ is metric tensor on $\mathbb{E}$.

Here X is point on the shape $\mathcal{S}$, $\lambda : \mathcal{S} \times [t_0, t_1] \to \mathbb{E}$ is a motion, and G, $\mathfrak{T}$ are, respectively, time-dependent metric and torsion.

22°. Micromorphic body. The case of micromorphic continuum can be considered in a similar manner. One starts with some submanifold $VM_I \subset V\mathbb{E}$, the substructure of intermediate shape $V\mathbb{S}_I$, pure from any geometry. Meanwhile, in contrast to the previous considerations, one has two tensor fields of local deformations: $\boldsymbol{H} : M_I \to \mathrm{Lin}(\mathbb{V};\, \mathbb{V})$ and $\boldsymbol{D} : M_I \to \mathrm{Lin}(\mathbb{V};\, \mathrm{Lin}(\mathbb{F};\, \mathbb{F}))$. Here M_I is the base manifold. Moreover, we suppose that model fiber $\mathbb{F}$ has inner product structure, which, in turn, induces inner product (:) on $\mathrm{Lin}(\mathbb{F};\, \mathbb{F})$.

Using natural isomorphism $T_X M_I \cong \mathbb{V}$, we define two Riemannian metrics on M_I. One is the Riemannian metric G, introduced by formula (5.5). The second field $\mathsf{G}^{\mathbb{F}} : M_I \to T^*M_I \otimes T^*M_I$ is defined by the following equality:

$$\mathsf{G}^{\mathbb{F}}_X[\mathsf{u},\, \mathsf{v}] = \boldsymbol{D}_X[\mathsf{u}] : \boldsymbol{D}_X[\mathsf{v}], \tag{5.8}$$

for all $X \in M_I$ and $\mathsf{u}, \mathsf{v} \in T_X M_I$. The metric $\mathsf{G}^{\mathbb{F}}$ is a metric over orientations, i. e., with each point $X \in M_I$ it associates a machine, that measures lengths of orientations associated with point X and angles between them.

Similarly to metrics, we introduce two connections on M_I. One is affine connection ∇ on M_I with connection functions defined as (5.6). Its torsion tensor $\mathfrak{T}$ characterizes the inhomogeneity within particles. Using metric field $\mathsf{G}^{\mathbb{F}}$ defined by (5.8), one can induce Levi-Civita connection $\nabla^{\mathbb{F}}$ on M_I. Its curvature tensor $\mathfrak{R}^{\mathbb{F}}$ characterizes the inhomogeneity within micro-particles. Thus, instead of (5.7), one arrives at the following Lagrangian density:

$$\mathscr{L} = \mathscr{L}(X,\, t,\, \lambda,\, T\lambda,\, \dot{\lambda},\, \mathsf{G},\, \dot{\mathsf{G}},\, \mathfrak{T},\, \dot{\mathfrak{T}},\, \mathsf{G}^{\mathbb{F}},\, \dot{\mathsf{G}}^{\mathbb{F}},\, \mathfrak{R}^{\mathbb{F}},\, \dot{\mathfrak{R}}^{\mathbb{F}}), \tag{5.9}$$

where $X \in M_I$ and mapping $\lambda : M_I \times [t_0, t_1] \to \mathbb{E}$ corresponds to motion.

Conclusions

We point out the following positions of the classical method, which are specific for oriented solids and shell-like solids with incompatible deformations.

1. The action is formulated with respect to intermediate shape, which is related with the continuous family of local stress free shapes. This brings additional field arguments, namely implants (with translation and microdeformational character), to the Lagrangian density.

2. It is possible to formulate the action with respect to single stress-free shape, but, generally, this shape will be non-Euclidean.

3. The principal difference between models for micropolar, micromorphic and shell-like solids is encoded in the structure of bundles, that are used for enhanced manifold definition.

4. Most general structure that characterize embedding of smooth material bundle into smooth physical bundle is too wide for all conventional models, because it takes into account a continuum of possible orientations for each material point. Meanwhile one can derive from it conventional models for oriented solids and shells by constructing specific sections over the bundles.

Appendix

23°. Metric and connection on body manifold. Let us consider in more detail what gives the use of metric and connection in the structure of body manifold. Riemannian metric $\mathsf{g}_B : \mathfrak{B} \to T^*\mathfrak{B} \otimes T^*\mathfrak{B}$ satisfies the following defining properties at each point $\mathfrak{X} \in \mathfrak{B}$ [25]:

- for all tangent vectors $\mathsf{u}, \mathsf{v} \in T_{\mathfrak{X}}\mathfrak{B}$ one has that $\mathsf{g}_B|_{\mathfrak{X}}(\mathsf{u},\, \mathsf{v}) = \mathsf{g}_B|_{\mathfrak{X}}(\mathsf{v},\, \mathsf{u})$;

- for each vector $\mathsf{u} \in T_{\mathfrak{X}}\mathfrak{B}$, $\mathsf{g}_B|_{\mathfrak{X}}(\mathsf{u},\, \mathsf{u}) \geqslant 0$;

- for any vector $\mathsf{u} \in T_{\mathfrak{X}}\mathfrak{B}$, $\mathsf{g}_B|_{\mathfrak{X}}(\mathsf{u},\, \mathsf{u}) = 0$ iff $\mathsf{u} = \mathsf{0}$.

Thus, the bilinear form $(\mathsf{u},\, \mathsf{v}) \mapsto \mathsf{g}_B|_{\mathfrak{X}}(\mathsf{u},\, \mathsf{v})$ is an inner product on tangent space $T_{\mathfrak{X}}\mathfrak{B}$. One can measure lengths of tangent vectors and angles between them. After embedding $\varkappa : \mathfrak{B} \to \mathbb{E}$ to physical manifold one can compare metric information on $\mathfrak{B}$ with metric information on the image $\varkappa(\mathfrak{B})$ of the embedding. In the framework of conventional elasticity this description gives nothing new, since one deals not with body, but with some fixed privileged subspace of $\mathbb{E}$, shape of the body. Usually it is assumed that this shape is stress-free.

Meanwhile, for solid with defects one cannot find such the shape in Euclidean physical space. Geometric methodology, developed in the papers [22; 23; 36–38] suggests to fix the issue by using non-Euclidean shapes, modeled on the body manifold. Then metric g_B acquires the certain sense: it gives reference (in stress-free state) lengths and angles of infinitesimal material fibers, formalized as tangent vectors. At the same time, one needs an extra information: to formulate balance equations on the body manifold one needs certain parallel transport rule. This rule can be established be means of the affine connection ∇_B, which to any pair (u, v) of vector fields on $\mathfrak{B}$ assigns vector field $(\nabla_B)_{\mathsf{u}}\mathsf{v}$ on the same body manifold. The following requirements are satisfied [39]:

- $(\nabla_B)_{\mathsf{u}+\mathsf{v}}\mathsf{w} = (\nabla_B)_{\mathsf{u}}\mathsf{w} + (\nabla_B)_{\mathsf{v}}\mathsf{w}$,
- $(\nabla_B)_{f\mathsf{u}}\mathsf{v} = f(\nabla_B)_{\mathsf{u}}\mathsf{v}$,
- $(\nabla_B)_{\mathsf{u}}(\mathsf{v}+\mathsf{w}) = (\nabla_B)_{\mathsf{u}}\mathsf{v} + (\nabla_B)_{\mathsf{u}}\mathsf{w}$,
- $(\nabla_B)_{\mathsf{u}}(f\mathsf{v}) = f(\nabla_B)_{\mathsf{u}}\mathsf{v} + (\mathsf{u}f)\mathsf{v}$,

for vector fields u, v, $\mathsf{w} : \mathfrak{B} \to T\mathfrak{B}$ and scalar function $f : \mathfrak{B} \to \mathbb{R}$. Affine connection endows body with some geometry. Each geometry is characterized by tensor fields of torsion $\mathfrak{T}$, curvature $\mathfrak{R}$ and nonmetricity $\mathfrak{Q}$ [24]. The correspondence to each space is shown on Table 2, where the symbol $\circ$ illustrates that the corresponding field vanishes, while $\bullet$ designates that field takes nonzero values.

Table 2

Correspondence between geometries and tensor fields of torsion, curvature and nonmetricity

Geometry	Torsion ($\mathfrak{T}$)	Curvature ($\mathfrak{R}$)	Nonmetricity ($\mathfrak{Q}$)
Riemann	∘	•	∘
Weitzenböck	•	∘	∘
Weyl	∘	∘	•

24°. Enhanced kinematics for vector bundles. The mapping ε from (3.3) has general form in the case of arbitrary fiber bundles. Meanwhile, vector bundle and principal bundle structures induce some particular properties of ε. Consider the case of vector bundle first. Physical bundle is represented by structure $(V\mathbb{E}, \mathbb{E}, \pi_{\mathbb{E}}, \mathbb{F})$, which is vector bundle of rank m, like material bundle. The enhanced configuration $V\varkappa : V\mathfrak{B} \to V\mathbb{E}$ and enhanced deformation $V\gamma : V\mathbb{S}_R \to V\mathbb{S}$ are related to conventional configuration $\varkappa : \mathfrak{B} \to \mathbb{E}$ and conventional deformation $\gamma : \mathbb{S}_R \to \mathbb{S}$ according to the general formulae (3.1) and (3.2):

$$\varkappa \circ \pi_{\mathfrak{B}} = \pi_{\mathbb{E}} \circ V\varkappa, \quad \gamma \circ \pi_{\mathbb{E}} = \pi_{\mathbb{E}} \circ V\gamma.$$

Moreover, it is required that mappings $V\varkappa$ and $V\gamma$ are compatible with vectorial structure of the bundle[18] [25]: for any $\mathfrak{X} \in \mathfrak{B}$ the restriction $V\varkappa|_{V\mathfrak{B}_{\mathfrak{X}}} : V\mathfrak{B}_{\mathfrak{X}} \to V\mathbb{E}_{\varkappa(\mathfrak{X})}$ to fibers of $V\mathfrak{B}$ is linear map, and for any $X \in \mathbb{S}_R$ the restriction[19] $V\gamma|_{V\mathbb{S}_{R;\,X}} : V\mathbb{S}_{R;\,X} \to V\mathbb{S}_{\gamma(X)}$ to fibers of $V\mathbb{S}_R$ is also a linear map.

Introduce natural coordinates $(X^1, ..., X^n; U^1, ..., U^m)$ on the enhanced reference shape $V\mathbb{S}_R$ and natural coordinates $(x^1, ..., x^n; u^1, ..., u^m)$ on the enhanced actual shape $V\mathbb{S}$. Let $\tilde{\gamma} : (X^1, ..., X^n) \mapsto (x^1, ..., x^n)$ be coordinate representation of conventional deformation γ. Then bundle structure gives the following equality for coordinate representation $\widetilde{V\gamma} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ of $V\gamma$:

$$\widetilde{V\gamma}(X^1, ..., X^n; U^1, ..., U^m) = (\tilde{\gamma}(X^1, ..., X^n); \tilde{\boldsymbol{\mathcal{M}}}(X^1, ..., X^3; U^1, ..., U^m)),$$

where $\tilde{\boldsymbol{\mathcal{M}}} : \mathbb{R}^{m+n} \to \mathbb{R}^m$ gives coordinate representation of actual orientations. The linearity condition imposed on enhanced deformation imply that partial map $\tilde{\boldsymbol{\mathcal{M}}}(X^1, ..., X^n, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is linear.

Consider particular case, when physical vector bundle is trivial, i. e., $V\mathbb{E}$ is diffeomorphic to product space $\mathbb{E} \times \mathbb{F}$ via local trivialization Φ. Then, since vector bundle structures are trivial, the enhanced deformation is the mapping $V\gamma : \mathbb{S}_R \times \mathbb{F} \to \mathbb{S} \times \mathbb{F}$, which values are of the form

$$(x, u) = V\gamma(X, U) = (\gamma(X), \boldsymbol{\mathcal{M}}(X, U)), \tag{5.10}$$

where assignment $\boldsymbol{\mathcal{M}} : \mathbb{S}_R \times \mathbb{F} \to \mathbb{F}$ returns actual orientations and it is such that its partial map $\boldsymbol{\mathcal{M}}(X, \cdot\,\cdot) : \mathbb{F} \to \mathbb{F}$, $\boldsymbol{\mathcal{M}}(X, \cdot) : U \mapsto \boldsymbol{\mathcal{M}}(X, U)$ is linear. Thus, in the case of trivial vector bundle we obtain that deformation of solid with extra degrees of freedom is characterized by conventional deformation $\gamma : \mathbb{S}_R \to \mathbb{S}$ and so-called microdeformation tensor $\boldsymbol{\mathcal{M}} : \mathbb{S}_R \times \mathbb{F} \to \mathbb{F}$.

[18] If not so, we won't obtain "nice" properties of the map ε.

[19] Here, as in general case of fiber bundle, we induce vector bundle structures on shapes: $(V\mathbb{S}_R, \mathbb{S}_R, \pi_{\mathbb{E}}|_{V\mathbb{S}_R}, \mathbb{F})$, and $(V\mathbb{S}, \mathbb{S}, \pi_{\mathbb{E}}|_{V\mathbb{S}}, \mathbb{F})$.

25°. Enhanced kinematics for principal bundles. Finally, we equip physical space $\mathbb{E}$ with principal bundle structure $(P\mathbb{E}, \mathbb{E}, \pi_{\mathbb{E}}, G, \top, \lhd_{\mathbb{E}})$, which structure group G is similar to the structure group of material principal bundle $(P\mathfrak{B}, \mathfrak{B}, \pi_{\mathfrak{B}}, G, \top, \lhd_{\mathfrak{B}})$. For further reasonings it would be convenient to denote coordinate tuples on base and structure group by putting tilde above. In particular, $\widetilde{\mathfrak{X}} = (\mathfrak{X}^1, ..., \mathfrak{X}^n)$ and $\widetilde{\mathfrak{g}} = (\mathfrak{g}^1, ..., \mathfrak{g}^m)$.

By enhanced configuration of enhanced body $P\mathfrak{B}$ to enhanced physical space $P\mathbb{E}$ we mean a smooth mapping $P\varkappa : P\mathfrak{B} \to P\mathbb{E}$, that satisfies the equivariance condition [29]:

$$\forall p \in P\mathfrak{B}\, \forall g \in G : \; P\varkappa(p \lhd_{\mathfrak{B}} g) = P\varkappa(p) \lhd_{\mathbb{E}} g. \tag{5.11}$$

In other words, $P\varkappa$ preserves principal bundle structure. In particular, it preserves layers and by this reason it induces embedding $\varkappa : \mathfrak{B} \to \mathbb{E}$, conventional configuration, such that the equality (3.1) holds.

Relation (3.1) induces representation of enhanced configuration $P\varkappa$ in natural coordinates:

$$\widetilde{P\varkappa}(\widetilde{\mathfrak{X}};\, \widetilde{\mathfrak{g}}) = (\widetilde{\varkappa}(\widetilde{\mathfrak{X}});\, \widetilde{\tau}(\widetilde{\mathfrak{X}};\, \widetilde{\mathfrak{g}})),$$

where $\widetilde{\varkappa} : \widetilde{\mathfrak{X}} \mapsto \widetilde{x}$ is coordinate representation of conventional configuration $\varkappa$, while $\widetilde{\tau} : (\widetilde{\mathfrak{X}}; \widetilde{\mathfrak{g}}) \mapsto \widetilde{\mathfrak{g}}$ corresponds to configuration of orientations. Moreover, equivariance condition (5.11) gives

$$\tau(\widetilde{p \lhd_{\mathfrak{B}}} \widetilde{h}) = \tau(\widetilde{p}) \widetilde{\top} \widetilde{h}.$$

Let $P\varkappa_R$, $P\varkappa : P\mathfrak{B} \to P\mathbb{E}$ be enhanced configurations of enhanced body $P\mathfrak{B}$, and let $\varkappa_R$, $\varkappa : \mathfrak{B} \to \mathbb{E}$ be corresponding conventional configurations of the body manifold $\mathfrak{B}$. Then images $P\mathbb{S}_R = P\varkappa_R(P\mathfrak{B})$ and $P\mathbb{S} = P\varkappa(P\mathfrak{B})$ correspond to enhanced shapes of $P\mathfrak{B}$ in enhanced physical space $P\mathbb{E}$. The composition $P\gamma = \widehat{P\varkappa} \circ \widehat{P\varkappa_R}^{-1} : P\mathbb{S}_R \to P\mathbb{S}$ of enhanced configurations correspond to enhanced deformation in enhanced physical space $P\mathbb{E}$, while $\gamma = \widehat{\varkappa} \circ \widehat{\varkappa}_R^{-1} : \mathbb{S}_R \to \mathbb{S}$ is conventional deformation of reference set of places $\mathbb{S}_R$ to the actual one, $\mathbb{S}$.

In natural coordinates on enhanced shapes the coordinate representation of enhanced deformation has the form

$$\widetilde{P\gamma}(\widetilde{X};\, \widetilde{G}) = (\widetilde{\gamma}(\widetilde{X});\, \widetilde{\varepsilon}(\widetilde{X};\, \widetilde{G})).$$

Here $\widetilde{\gamma} : \widetilde{X} \mapsto \widetilde{x}$ is coordinate representation of γ, while $\widetilde{\varepsilon} : (\widetilde{X};\, \widetilde{X}) \mapsto \widetilde{g}$ returns actual orientations.

It can be shown that $P\gamma$ is equivariant, i. e., it satisfies the property similar to (3.1). Then the mapping $\widetilde{\varepsilon}$ has the "homogeneity" property with respect to elements of G:

$$\widetilde{\varepsilon}(\widetilde{X};\, \widetilde{G} \widetilde{\top} \widetilde{H}) = \widetilde{\varepsilon}(\widetilde{X};\, \widetilde{G}) \widetilde{\top} \widetilde{H}.$$

Suppose that G can be covered by one chart. Then, if set $\Psi(\widetilde{X}) := \widetilde{\varepsilon}(\widetilde{X};\, \widetilde{1})$, where $\widetilde{1}$ is coordinate representation of identity of G, one obtains $\widetilde{\varepsilon}(\widetilde{X};\, \widetilde{G}) = \Psi(\widetilde{X}) \widetilde{\top} \widetilde{G}$, that gives

$$\widetilde{P\gamma}(\widetilde{X};\, \widetilde{G}) = (\widetilde{\gamma}(\widetilde{X});\, \Psi(\widetilde{X}) \widetilde{\top} \widetilde{G}).$$

Supposing that physical principal bundle is trivial[20], i. e., $P\mathbb{E} = \mathbb{E} \times G$, one obtains that enhanced deformation $P\gamma : \mathbb{S}_R \times G \to \mathbb{S} \times G$ can be expressed in the form

$$P\gamma(X,\, g) = (\gamma(X),\, \varepsilon(X,\, g)),$$

where $\gamma : \mathbb{S}_R \to \mathbb{S}$ is conventional deformation, and $\varepsilon : \mathbb{S}_R \times G \to G$ returns actual orientations. If we put $\Psi(X) := \varepsilon(X,\, 1)$, where 1 is the identity of G, then, since $\varepsilon(X,\, g) = \Psi(X) \top g$, one gets

$$P\gamma(X,\, g) = (\gamma(X),\, \Psi(X) \top g).$$

Thus, deformation of oriented solid is characterized by mappings $\gamma : \mathbb{S}_R \to \mathbb{S}$ and $\Psi : \mathbb{S}_R \to G$.

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НЕЛИНЕЙНЫЕ ДИНАМИЧЕСКИЕ УРАВНЕНИЯ ДЛЯ УПРУГИХ МИКРОМОРФНЫХ ТЕЛ И ОБОЛОЧЕК. ЧАСТЬ I[21]

АННОТАЦИЯ

В настоящей статье развивается общий подход к выводу нелинейных уравнений движения для деформируемых твердых тел, материальные точки которых обладают дополнительными степенями свободы. Характерной чертой этого подхода является учет несовместных деформаций, которые могут возникнуть в теле из-за распределенных дефектов или в результате некоторого процесса, например наращивания или ремоделирования. Математический формализм основан на принципе наименьшего действия и нетеровых симметриях. Особенность такого формализма заключается в формальном описании отсчетной конфигурации тела, которое в случае несовместных деформаций следует рассматривать или как непрерывное семейство

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форм, или как некоторую форму, вложенную в неевклидово пространство. Хотя общий подход дает уравнения для деформируемых твердых тел типа Коссера, микроморфных тел и оболочек, последние существенно отличаются по формальному описанию расширенной геометрической структуры, для которой необходимо определить интеграл действия. Это различие подробно обсуждается.

Ключевые слова: нелинейная динамика; микрополярные и микроморфные тела; оболочки; конечные деформации; несовместность деформаций; неевклидова отсчетная форма; расслоение; расширенные материальное и физическое многообразия; наименьшее действие; нетеровы симметрии; уравнения поля; законы сохранения.

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Информация о конфликте интересов: авторы и рецензенты заявляют об отсутствии конфликта интересов.

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