DOI: 10.18287/2541-7525-2021-27-1-29–43

Submited: 15.01.2021
Revised: 17.02.2021
Accepted: 28.02.2021

E. Providas
University of Thessaly, Larissa, Greece
E-mail: providas@uth.gr. ORCID: https://orcid.org/0000-0002-0675-4351
L.S. Pulkina
Samara National Research University, Samara, Russian Federation
E-mail: louise@samdiff.ru. ORCID: https://orcid.org/0000-0001-7947-612
I.N. Parasidis
University of Thessaly, Larissa, Greece
E-mail: paras@teilar.gr. ORCID: https://orcid.org/0000-0002-7900-9256

FACTORIZATION OF ORDINARY AND HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS IN A BANACH SPACE

ABSTRACT

The solvability condition and the unique exact solution by the universal factorization (decomposition) method for a class of the abstract operator equations of the type

$$B_1u = \mathcal{A}u - S\Phi(A_0u) - GF(\mathcal{A}u) = f, \quad u \in D(B_1),$$

where $\mathcal{A}, A_0$ are linear abstract operators, G, S are linear vectors and Φ, F are linear functional vectors is investigagted. This class is useful for solving Boundary Value Problems (BVPs) with Integro-Differential Equations (IDEs), where $\mathcal{A}, A_0$ are differential operators and $F(\mathcal{A}u), \Phi(A_0u)$ are Fredholm integrals. It was shown that the operators of the type B_1 can be factorized in the some cases in the product of two more simple operators B_G, B_{G_0} of special form, which are derived analytically. Further the solvability condition and the unique exact solution for $B_1u = f$ easily follow from the solvability condition and the unique exact solutions for the equations $B_Gv = f$ and $B_{G_0}u = v$.

Key words: correct operator; factorization (decomposition) method; Fredholm integro-differential equations; initial problem; nonlocal boundary value problem with integral boundary conditions.

Citation. Providas E., Pulkina L.S., Parasidis I.N. Factorization of ordinary and hyperbolic integro-differential equations with integral boundary conditions in a Banach space. *Vestnik Samarskogo universiteta. Estestvennonauchnaia seriia = Vestnik of Samara University. Natural Science Series*, 2021, vol. 27, no. 1, pp. 29–43. DOI: http://doi.org/10.18287/2541-7525-2021-27-1-29–43.

Information about the conflict of interests: authors and reviewers declare no conflict of interests.

Efthimios Providas — Candidate of Technical Sciences, associate professor, University of Thessaly, Larissa, 41110, Greece.

Ludmila Stepanovna Pulkina — Doctor of Physical and Mathematical Sciences, professor, Department of Differential Equations and Control Theory, Samara National Research University, 34, Moskovskoye shosse, 443086, Russian Federation.

Ioannis Nestorios Parasidis — Candidate of Technical Sciences, associate professor, University of Thessaly, Larissa, 41110, Greece.

1. Initial Position

Integro-differential equations play an important role in characterizing many physical, biological, social and engineering problems and are often solved by factorization (decomposition) methods. The factorization methods have applications in biology, ecology, population dynamics, mathematics of financial derivatives, quantum physics, hydrodynamics, gas dynamics, in transport theory, electromagnetic theory, mechanics and chemistry [1–10]. Factorization Methods successfully are used in pure mathematics for solving linear and nonlinear ordinary and partial differential and Volterra-Fredholm integro-differential equations, integro-differential equations of fractional order, fuzzy Volterra-Fredholm integral equations and delay differential equations [11–20]. There are well-known decomposition (factorization) methods: Domain decomposition method, the natural transform decomposition method, the Adomian decomposition method, Modified Adomian decomposition method and the Combined Laplace transform-Adomian decomposition method, which use the so-called Adomyan polynomials or iterations to obtain an n-term approximation of solution, whereas the proposed in this paper factorization method gives the unique exact solution in the closed form. Furthermore it is universal because can be applied in the investigation of Fredholm integro-differential equations, both ordinary and partial.

There are many papers are devoted to investigation of the uniqueness of the solution to nonlocal boundary value problems with integral boundary conditions for hyperbolic differential equation [21–25]. Finding of the exact solution in the general case is the difficult task. We by the universal factorization method find the solvability condition and a unique solution to a nonlocal BVP with integral boundary conditions for Fredholm ordinary integro-differential and integro-hyperbolic differential equations of the type $B_1u = f$. It is the aim of this paper to reappraise the factorization method for integro-differential equations of type $B_1u = f$. This paper is a generalization of the article [26], where by factorization method were stydied the solvability condition and a unique solution to the correct self-adjoint abstract equation of the type $B_1u = f$ in terms of a Hermitian matrix in a Hilbert space.

The quadratic factorization methods was applied to some BVPs with integro-differential equations in the case of a Banach space in [27–29].

It is well known that the class of the operators which can be factorized as a superposition of two more simple operators is not wide. But if the operator can be factorized, then the solvability condition and the solution of the given problem are essentially simpler than in the general case without factorization. The paper is organized as follows. In Section 2 we develop the theory for the solution of the problem $B_1x = f$ when $B_1 = BB_0$. Further by factorization method we solve a nonlocal boundary value problem with integral boundary conditions for Fredholm integro-hyperbolic differential equation. Finally, we give two examples of integro-differential equations demonstrating the power and usefulness of the methods presented.

Throughout this paper we use the following terminology and notation. By X, Y we denote the complex Banach spaces and by X^* the adjoint space of X, i.e. the set of all complex-valued linear and bounded functionals on X. We denote by $f(u)$ the value of f on $u \in X$. We write $D(A)$ and $R(A)$ for the domain and the range of the operator $A : X \to Y$, respectively. An operator A_2 is said to be an *extension* of an operator A_1, or A_1 is said to be a *restriction* of A_2, in symbol $A_1 \subset A_2$, if $D(A_2) \supseteq D(A_1)$ and $A_1x = A_2x$, for all $x \in D(A_1)$. An operator $A : X \to Y$ is said to be *injective or uniquelly solvable* if for all $u_1, u_2 \in D(A)$ such that $Au_1 = Au_2$, follows that $u_1 = u_2$. Remind that a linear operator A is injective if and only if $\ker A = \{0\}$. An operator $A : X \to Y$ is called *surjective* or *everywhere solvable* if $R(A) = Y$. The operator $A : X \to Y$ is called *bijective* if A is both injective and surjective. Lastly, A is said to be *correct* if A is bijective and its inverse A^{-1} is bounded on Y. If $g_i \in X$ and $\Psi_i \in X^*, i = 1, \ldots, m$, then we denote by $g = (g_1, \ldots, g_m)$, $\Psi = \mathrm{col}(\Psi_1, \ldots, \Psi_m)$ and $\Psi(u) = \mathrm{col}(\Psi_1(u), \ldots, \Psi_m(u))$ and write $g \in X_m$, $\Psi \in X_m^*$. We will denote by $\Psi(g)$ the $m \times m$ matrix whose i,j-th entry $\Psi_i(g_j)$ is the value of functional Ψ_i on element g_j. Note that $\Psi(gC) = \Psi(g)C$, where C is a $m \times k$ constant matrix. We will also denote by 0_m the zero and by I_m the identity $m \times m$ matrices. By $\mathbf{0}$ we will denote the zero column vector.

2. Factorization of integro-differential equations in a Banach space

We remind first the following Theorem 1 from [29].

Theorem 2.1. *Let A be a bijective operator on a Banach space X, the components of the vectors $G = (g_1, ..., g_m)$, $F = col(F_1, ...F_m)$ arbitrary elements of X and X^*, respectively and the operator $B_G : X \to X$ be defined by*

$$B_Gu = Au - GF(Au) = f, \quad D(B_G) = D(A), \quad f \in X. \tag{2.1}$$

Then the following statements are true:

(i) The operator B_G is bijective on X if and only if

$$\det L = \det[I_m - F(G)] \neq 0, \tag{2.2}$$

and the unique solution to boundary value problem (2.1), for any $f \in X$, is given by the formula

$$u = B_G^{-1} f = A^{-1} f + A^{-1} G[I_m - F(G)]^{-1} F(f). \tag{2.3}$$

(ii) If in addition the operator A is correct, then B_G is correct.

Now, by using the above theorem we prove the following theorem which is useful for solving integro-differential equations by factorization method.

Theorem 2.2. *Let X be a Banach space, the vectors $G_0 = (g_1^{(0)}, ..., g_m^{(0)})$, $G = (g_1, ..., g_m)$, $S = (s_1, ..., s_m) \in X^m$, the components of the vectors $F = col(F_1, ..., F_m)$ and $\Phi = col(\phi_1, ..., \phi_m)$ belong to X^* and the operators $B_{G_0}, B_G, B_1 : X \to X$ defined by*

$$B_{G_0} u = A_0 u - G_0 \Phi(A_0 u) = f, \qquad D(B_{G_0}) = D(A_0), \tag{2.4}$$

$$B_G u = Au - GF(Au) = f, \qquad D(B_G) = D(A), \tag{2.5}$$

$$B_1 u = AA_0 u - S\Phi(A_0 u) - GF(AA_0 u) = f, \quad D(B_1) = D(AA_0) \tag{2.6}$$

where A_0 and A are linear correct operators on X and $G_0 \in D(A)^m$. Then the following statements are satisfied:
(i) If

$$S \in R(B_G)^m \quad \text{and} \quad S = B_G G_0 = AG_0 - GF(AG_0), \tag{2.7}$$

then the operator B_1 can be factorised in $B_1 = B_G B_{G_0}$.
(ii) If in addition the components of the vector $\Phi = (\Phi_1, ..., \Phi_m)$ are linearly independent elements of X^* and the operator B_1 can be factorised in $B_1 = B_G B_{G_0}$, then (2.7) is fulfilled.
(iii) If the operator B_1 can be factorised in $B_1 = B_G B_{G_0}$, then B_1 is correct if and only if the operators B_{G_0} and B_G are correct which means that

$$\det L_0 = \det[I_m - \Phi(G_0)] \neq 0 \quad \text{and} \quad \det L = \det[I_m - F(G)] \neq 0. \tag{2.8}$$

(iv) If the operator B_1 has the factorization in $B_1 = B_G B_{G_0}$ and is correct, then the unique solution of (2.6) is

$$\begin{aligned} u = B_1^{-1} f = \; & A_0^{-1} A^{-1} f + \left[A_0^{-1} A^{-1} G + A_0^{-1} G_0 L_0^{-1} \Phi(A^{-1} G)\right] \times \\ & \times L^{-1} F(f) + A_0^{-1} G_0 L_0^{-1} \Phi(A^{-1} f). \end{aligned} \tag{2.9}$$

Proof. (i) Taking into account that $G_0 \in D(A)^m$ and (2.4)–(2.6) we get

$$\begin{aligned} D(B_G B_{G_0}) & = \{u \in D(B_{G_0}) : \, B_{G_0} u \in D(B_G)\} = \\ & = \{u \in D(A_0) : \, A_0 u - G_0 \Phi(A_0 u) \in D(A)\} = \\ & = \{u \in D(A_0) : \, A_0 u \in D(A)\} = D(AA_0) = D(B_1). \end{aligned}$$

So $D(B_1) = D(B_G B_{G_0})$. Let $y = B_{G_0} u$. Then for each $u \in D(AA_0)$ since (2.5) and (2.4) we have

$$\begin{aligned} B_G B_{G_0} u & = B_G y = Ay - GF(Ay) = \\ & = A[A_0 u - G_0 \Phi(A_0 u)] - GF\left(A[A_0 u - G_0 \Phi(A_0 u)]\right) = \\ & = AA_0 u - AG_0 \Phi(A_0 u) - GF(AA_0 u) + GF(AG_0)\Phi(A_0 u) = \\ & = AA_0 u - [AG_0 - GF(AG_0)]\Phi(A_0 u) - GF(AA_0 u) = \\ & = AA_0 u - B_G G_0 \Phi(A_0 u) - GF(AA_0 u), \end{aligned} \tag{2.10}$$

where the relation $B_G G_0 = AG_0 - GF(AG_0)$ follows from (2.5) if instead of u we take G_0. By comparing (2.10) with (2.6), it is easy to verify that $B_1 u = B_G B_{G_0} u$ for each $u \in D(AA_0)$ if a vector S satisfies (2.7).
(ii) Let the operator B_1 can be factorized in $B_1 = B_G B_{G_0}$. Then by comparing (2.10) with (2.6) we obtain

$$(B_G G_0 - S)\Phi(A_0 u) = 0. \tag{2.11}$$

Because of the correctness of operators A, A_0 and the linear independence of $\Phi_1, ..., \Phi_m$, there exists a system $u_1, ..., u_m \in D(AA_0)$ such that $\Phi(A_0 u_0) = I_m$ where $u_0 = (u_1, ..., u_m)$. By substituting $u = u_0$ into (2.11) we get $S = B_G G_0$. Hence $S \in R(B_G)^m$ and $S = B_G G_0 = AG_0 - GF(AG_0)$.
(iii) Let the operator B_1 be defined by (2.6) where $S = B_G G_0$. Then Equation (2.6) can be equivalently represented in matrix form:

$$B_1 u = AA_0 u - (B_G G_0, G) \begin{pmatrix} \Phi(A^{-1} AA_0 u) \\ F(AA_0 u) \end{pmatrix} = f \tag{2.12}$$

or

$$B_1u = \mathcal{A}u - \mathcal{G}\mathcal{F}(\mathcal{A}u) = f, \quad D(B_1) = D(\mathcal{A}), \tag{2.13}$$

where $\mathcal{A} = AA_0$, $\mathcal{G} = (B_G G_0, G)$, $\mathcal{F} = col(\hat{\Phi}, F)$, $\mathcal{F}(v) = \begin{pmatrix} \hat{\Phi}(v) \\ F(v) \end{pmatrix} = \begin{pmatrix} \Phi(A^{-1}v) \\ F(v) \end{pmatrix}$. Notice that the operator $\mathcal{A} = AA_0$ is correct, because of A and A_0 are the correct operators and that the functional vector $\mathcal{F}$ is bounded, since the vector $\hat{\Phi}$ is bounded as a superposition of a bounded functional Φ and a bounded operator A^{-1}. Then we apply Theorem 2.1. By this theorem the operator B_1 is correct if and only if

$$\det L_1 = \det[I_{2m} - \mathcal{F}(\mathcal{G})] = \det\left[\begin{pmatrix} I_m & 0_m \\ 0_m & I_m \end{pmatrix} - \begin{pmatrix} \hat{\Phi}(B_G G_0) & \hat{\Phi}(G) \\ F(B_G G_0) & F(G) \end{pmatrix}\right] =$$

$$= \det\begin{pmatrix} I_m - \hat{\Phi}(AG_0 - GF(AG_0)) & -\hat{\Phi}(G) \\ -[F(AG_0 - GF(AG_0))] & I_m - F(G) \end{pmatrix} =$$

$$= \det\begin{pmatrix} I_m - \Phi(G_0 - A^{-1}GF(AG_0)) & -\Phi(A^{-1}G) \\ -[F(AG_0 - GF(AG_0))] & I_m - F(G) \end{pmatrix} =$$

$$= \det\begin{pmatrix} I_m - \Phi(G_0) + \Phi(A^{-1}G)F(AG_0) & -\Phi(A^{-1}G) \\ -F(AG_0) + F(G)F(AG_0) & I_m - F(G) \end{pmatrix} \neq 0.$$

Multiplying from the left the elements of the second column by $F(AG_0)$ and adding to the corresponding elements of the first column of the determinant L_1, by Remark 1, [31] we get

$$\det L_1 = \det\begin{pmatrix} I_m - \Phi(G_0) & -\Phi(A^{-1}G) \\ 0_m & I_m - F(G) \end{pmatrix} = \det[I_m - \Phi(G_0)]\det[I_m - F(G)]$$
$$= \det L_0 \det L \neq 0.$$

So we proved that the operator B_1 is correct if and only if (2.8) is fulfilled.
(iv) Let $u \in D(AA_0)$ and $B_G B_{G_0} u = f$. By Theorem 2.1 (ii) since B_G, B_{G_0} are correct operators, we obtain

$$B_{G_0}u = B_G^{-1}f = A^{-1}f + A^{-1}GL^{-1}F(f),$$
$$u = B_{G_0}^{-1}\left(A^{-1}f + A^{-1}GL^{-1}F(f)\right).$$

In the last equation we denote by $g = A^{-1}f + A^{-1}GL^{-1}F(f)$. Bu using again Theorem 2.1 (ii), with A_0, G_0, Φ, L_0, in place of A, G, F, L respectively, we get

$$u = B_{G_0}^{-1}g = A_0^{-1}g + A_0^{-1}G_0L_0^{-1}\Phi(g) = A_0^{-1}\big(A^{-1}f + A^{-1}GL^{-1}F(f)\big)+$$
$$+A_0^{-1}G_0L_0^{-1}\Phi\left(A^{-1}f + A^{-1}GL^{-1}F(f)\right) = A_0^{-1}A^{-1}f + A_0^{-1}A^{-1}GL^{-1}F(f)+$$
$$+A_0^{-1}G_0L_0^{-1}\big[\Phi(A^{-1}f) + \Phi(A^{-1}G)L^{-1}F(f)\big]$$

which implies (2.9). The theorem is proved. □

The next theorem is useful for applications.

Theorem 2.3. *Let the space X and the vectors F, Φ be defined as in Theorem 2.2, the vectors $G = (g_1, ..., g_m)$, $S = (s_1, ..., s_m) \in X^m$ and the operator $B_1 : X \to X$ by*

$$B_1u = \mathcal{A}u - S\Phi(A_0u) - GF(\mathcal{A}u) = f, \quad x \in D(B_1) \tag{2.14}$$

where $A_0 : X \to X$ is a correct m-order differential operator and $\mathcal{A}$ is a n-order differential operator, $m < n$. Then the next statements are fulfilled:
(i) if there exist a $n - m$ order differential operator $A : X \to X$, such that

$$\mathcal{A} = AA_0, \quad D(B_1) = D(AA_0), \tag{2.15}$$

and a vector $G_0 \in D(A)$, satisfying

$$AG_0 - GF(AG_0) = S, \tag{2.16}$$

then the operator B_1 can be factorized into $B_1 = B_G B_{G_0}$, where B_{G_0} and B_G are given by (2.4) and (2.5) respectively, B_G is determined by A and G, F from (2.14), (2.15) and lastly, the operator B_{G_0} by A_0, Φ and G_0 from (2.14) and (2.16),
(ii) if there exists a bijective $n - m$ order differential operator $A : X \to X$, satisfying (2.15) and

$$\det L = \det[I_m - F(G)] \neq 0, \tag{2.17}$$

then the operator B_1 is factorized in $B_1 = B_G B_{G_0}$, where the operators B_{G_0}, B_G, A_0, A, the vectors G, F, Φ are determined as in (i) and

$$G_0 = A^{-1}S + A^{-1}GL^{-1}F(S). \tag{2.18}$$

(iii) if in addition to (ii) A is correct, then B_1 is correct if and only if

$$\det L_0 = \det[I_m - \Phi(G_0)] = \det[I_m - \Phi(\widehat{A}^{-1}S) - \Phi(\widehat{A}^{-1}G)L^{-1}F(S)] \neq 0, \tag{2.19}$$

and the problem (2.14)-(2.16) has the unique solution given by (2.9).

Proof (i) If there there exist a $n-m$ order differential operator A and a vector G_0 satisfying (2.15) and (2.16), then from (2.14) we get

$$B_1 u = AA_0 u - S\Phi(A_0 u) - GF(AA_0 u) = f, \quad u \in D(AA_0). \tag{2.20}$$

From (2.20) we take a triple of elements, the operator A and vectors G, F, and construct the operator B_G according to the formula (2.5). To determine the operator B_{G_0} by formula (2.4), we take from (2.20) the operator A_0 and the vector Φ, whereas as G_0 we take any solution G_0 of Equation (2.16). We proved in the previous theorem (i) that $D(B_G B_{G_0}) = D(AA_0) = D(B_1)$. Substituting (2.16) into (2.20), for every $u \in D(B_1)$ we get

$$\begin{aligned} B_1 u &= AA_0 u - [AG_0 - GF(AG_0)]\,\Phi(A_0 u) - GF(AA_0 u) = B_G A_0 u - B_G G_0 \Phi(A_0 u) = \\ &= B_G\,[A_0 u - G_0 \Phi(A_0 u)] = B_G B_{G_0} u. \end{aligned}$$

Thus $B_1 = B_G B_{G_0}$.
(ii) As in the proof of (i) we construct the operators B_G, B_{G_0}. By Theorem 2.1, since (2.17), the operator B_G is correct and Equation (2.16) can be presented by $B_G G_0 = S$. Then $G_0 = B_G^{-1} S$. The last equation by Corollary 2.1, implies the unique vector G_0 by (2.18). Further as in the proof of (i) we get the factorization $B_1 = B_G B_{G_0}$, where B_{G_0} is unique.
(iii) If (2.17), (2.18) hold true, then by statements (i), (ii), B_1 can be factorized in $B_1 = B_G B_{G_0}$. By Theorem 2.2 (iii), B_1 is correct if and only if (2.8) holds or, taking into account (2.17) and (2.18), if and only if $\det L_0 = \det[I_m - \Phi(G_0)] \neq 0$, or if and only if (2.19) is fulfilled. The last inequality immediately follows by substitution (2.18) into $\det L_0 = \det[I_m - \Phi(G_0)]$. Since B_1 is correct and factorized in $B_1 = B_G B_{G_0}$, by Theorem 2.2 (iv), we obtain the unique solution (2.9) to the problem (2.14)-(2.16). So the theorem is proved. □

Example 2.4. Let $u(x) \in C^2[0,1]$. Then the problem

$$\begin{gathered} u''(t) - t\int_0^1 tu'(t)dt - t^2\int_0^1 t^3 u''(t)dt = 2t+1, \\ u(0)+u(1)=0, \quad u'(0)-2u'(1)=0, \end{gathered} \tag{2.21}$$

is correct on $C[0,1]$ and its unique solution is given by the formula

$$u(t) = \frac{261377 - 665232t + 103608t^2 + 30080t^3 + 8790t^4}{207216}. \tag{2.22}$$

Proof. First we need to find the operators B_1, A, A_0 and check the condition $D(B_1) = D(AA_0)$. If we compare equation (2.21) with equation (2.14), (2.15), it is natural to take

$$B_1 u(t) = u''(t) - t\int_0^1 tu'(t)dt - t^2\int_0^1 t^3 u''(t)dt = 2t+1, \tag{2.23}$$

$$D(B_1) = \{u(t) \in C^2[0,1] : u(0)+u(1)=0, \quad u'(0)-2u'(1)=0\},$$

$$\mathcal{A}u = AA_0 u = u''(t),$$

$$D(B_1) = \{u(t) \in C^2[0,1] : u(0)+u(1)=0, \quad u'(0)-2u'(1)=0\}, \tag{2.24}$$

$$A_0 u(t) = u'(t), \quad D(A_0) = \{u(t) \in C^1[0,1] : u(0) = -u(1)\},$$

$$\Phi(A_0 u) = \int_0^1 tu'(t)dt, \quad F(AA_0 u) = \int_0^1 t^3 u''(t)dt, \tag{2.25}$$

$S = t$, $G = t^2$. Denote $A_0 u(t) = u'(t) = y(t) = y$. Then from (2.24) we have $y \in D(A)$, $AA_0 u = (u'(t))' = y'(t) = Ay(t)$, $y(0) - 2y(1) = 0$. So we proved that

$$Ay = y'(t), \quad D(A) = \{y(t) \in C^1[0,1] : y(0) - 2y(1) = 0\}.$$

Then by definition

$$\begin{aligned} D(AA_0) &= \{u(t) \in D(A_0) : A_0 u(t) \in D(A)\} = \\ &= \{u(t) \in C^1[0,1] :\ u(0) = -u(1),\ u'(t) \in C^1[0,1],\ u'(0) - 2u'(1) = 0\} = \\ &= \{u(t) \in C^2[0,1] : u(0)+u(1)=0, \quad u'(0)-2u'(1)=0\} = D(B_1). \end{aligned}$$

So $D(B_1) = D(AA_0)$. It is easy to verify that the operators A, A_0 are correct on $C[0,1]$ and for every $f(t) \in C[0,1]$ the corresponding inverse operators are defined by

$$A^{-1}f(t) = \int_0^t f(s)ds - 2\int_0^1 f(s)ds, \tag{2.26}$$

$$A_0^{-1}f(t) = \int_0^t f(s)ds - \tfrac{1}{2}\int_0^1 f(s)ds. \tag{2.27}$$

From (2.25) we have

$$\Phi(f) = \int_0^1 sf(s)ds, \quad F(f) = \int_0^1 s^3 f(s)ds. \tag{2.28}$$

Then $F(G) = \int_0^1 s^3 s^2 ds = \frac{1}{6}$, $F(S) = \int_0^1 s^3 s ds = \frac{1}{5}$,
$\det L = \det[I_m - F(G)] = 1 - 1/6 = 5/6$, $L^{-1} = 6/5$,

$$A^{-1}S = \int_0^t sds - 2\int_0^1 sds = \frac{t^2}{2} - 1, \quad A^{-1}G = \int_0^t s^2ds - 2\int_0^1 s^2 ds = \frac{t^3}{3} - \frac{2}{3},$$

$$G_0 = A^{-1}S + A^{-1}GL^{-1}F(S) = \frac{t^2}{2} - 1 + \left(\frac{t^3}{3} - \frac{2}{3}\right)\frac{6}{5}\frac{1}{5} = \frac{1}{50}(4t^3 + 25t^2 - 58).$$

Taking into account (2.28) we obtain

$$\Phi(G_0) = \frac{1}{50}\int_0^1 s(4s^3 + 25s^2 - 58)ds = -\frac{439}{1000}.$$

Since $\det L_0 = \det[I_m - \Phi(G_0)] = \frac{1439}{1000} \neq 0$ then, by Theorem 2.3 (iii), Problem (2.23) or (2.21) is correct. By (2.27) we calculate

$$A_0^{-1}G_0 = \frac{73}{150} - \frac{29t}{25} + \frac{t^3}{6} + \frac{t^4}{50}, \quad A_0^{-1}A^{-1}G = \frac{7}{24} - \frac{2t}{3} + \frac{t^4}{12}$$

and for $f(t) = 2t + 1$ by (2.26)-(2.28) we obtain

$$A^{-1}f = -4 + t + t^2, \quad A_0^{-1}A^{-1}f = \frac{19}{12} - 4t + \frac{t^2}{2} + \frac{t^3}{3}, \quad F(f) = \frac{13}{20}, \quad \Phi(A^{-1}f) = -\frac{17}{12}$$

Substituting these values into (2.9) we obtain the unique solution of (2.23), which is given by (2.22). □

Example 2.5. Let $u(x) \in C^3[0,1]$. Then the problem

$$u'''(x) - 8x^2\int_0^1 tu'(t)dt - (3x+1)\int_0^1 t^2u'''(t)dt = 2x^2 - 6x + 4, \tag{2.29}$$
$$u(0) = 2\int_0^1 u(t)dt, \quad u'(0) = -u'(1),\ u''(0) = -u''(1),$$

is uniquely solvable on $C[0,1]$ and its unique solution is given by

$$u(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}. \tag{2.30}$$

Proof. First we must determine the operators B_1, A and A_0. By comparing Problem (2.29) with (2.14) it is natural to take $X = C[0,1]$,

$$B_1u = u'''(x) - 8x^2\int_0^1 tu'(t)dt - (3x+1)\int_0^1 t^2u'''(t)dt = 2x^2 - 6x + 4, \tag{2.31}$$

$$D(B_1) = \{u(x) \in C^3[0,1] : u(0) = 2\int_0^1 u(t)dt, \quad u'(0) = -u'(1),\ u''(0) = -u''(1)\},$$

$$\Phi(A_0u) = \int_0^1\int_0^1 tu'(t)dtdy,\ F(AA_0u) = \int_0^1\int_0^1 t^2u'''(t)dtdy, \tag{2.32}$$
$$\mathcal{A}x = AA_0u(x) = u'''(x), \quad A_0u = u'(x).$$

Denote $v(x) = u'(x)$. Then $AA_0u(x) = u'''(x) = (u'(x))'' = Av(x) = v''(x)$. From boundary conditions (2.31) follws that $v(0) = u'(0) = -u'(1) = -v(1)$, $v'(0) = u''(0) = -u''(1) = -v'(1)$. So the operators A, A_0 are defined by

$$Av(x) = v''(x), \quad D(A) = \{v(x) \in C^2[0,1] :\ v(0) = -v(1),\ v'(0) = -v'(1)\},$$

$$A_0u(x) = u'(x), \quad D(A_0) = \{u(x) \in C^1[0,1] :\ u(0) = 2\int_0^1 u(x)dx\}.$$

Now we make sure that $D(B_1) = D(AA_0)$. Using the definition of the product operators we get

$$D(AA_0) = \{u(x) \in D(A_0) : A_0u \in D(A)\} = \{u(x) \in C^1[0,1] : u(0) =$$
$$= 2\int_0^1 u(x)dx,\ u'(x) \in C^2[0,1],\ u'(0) = -u'(1),\ u''(0) = -u''(1)\} = D(B_1).$$

Since $D(B_1) = D(AA_0)$, we can apply Theorem 2.3. It is easy to verify that the operators A and A_0 are correct and their inverse operators for all $f(t) \in C[0,1]$ are given by

$$A_0^{-1}f(x) = 2\int_0^1 (t-1)f(t)dt + \int_0^x f(t)dt, \tag{2.33}$$

$$A^{-1}f(x) = \frac{1}{2}\int_0^1 \left(t - x - \frac{1}{2}\right) f(t)dt + \int_0^x (x-t)f(t)dt. \tag{2.34}$$

By comparing again (2.31) with (2.14) it is natural to take $S = S(x) = 8x^2$, $G = G(x) = 3x+1$. From (2.32) we get

$$\Phi(f) = \int_0^1 tf(t)dt, \quad F(f) = \int_0^1 t^2 f(t)dt. \tag{2.35}$$

Let $\hat{f}(x) = A^{-1}f(x)$ and $AA_0u(x) = f(x)$. Then, since A_0, A are invertible, by means of (2.33) and (2.34) we have

$$u(x) = A_0^{-1}A^{-1}f(x) = A_0^{-1}\hat{f}(x) = 2\int_0^1 (t-1)\hat{f}(t)dt + \int_0^x \hat{f}(t)dt =$$
$$= 2\int_0^1 (t-1)\left[\frac{1}{2}\int_0^1 \left(s - t - \frac{1}{2}\right) f(s)ds + \int_0^t (t-s)f(s)ds\right] dt +$$
$$+ \int_0^x \left[\frac{1}{2}\int_0^1 \left(s - t - \frac{1}{2}\right) f(s)ds + \int_0^t (t-s)f(s)ds\right] dt.$$

Further using Fubini theorem we obtain

$$A_0^{-1}A^{-1}f(x) = -\frac{1}{12}\int_0^1 [3x^2 + 3x(1-2s) - 4s^3 + 12s^2 - 6s - 1]f(s)ds + \\ + \frac{1}{2}\int_0^x (x-s)^2 f(s)ds. \tag{2.36}$$

Using (2.36) for $f = f(x) = 2x^2 - 6x + 4$ and $G = 3x+1$ we get

$$A_0^{-1}A^{-1}f = -\frac{1}{12}\int_0^1 [3x^2 + 3x(1-2s) - 4s^3 + 12s^2 - 6s - 1](2s^2 - 6s + 4)ds + \tag{2.37}$$
$$+\frac{1}{2}\int_0^x (x-s)^2(2s^2 - 6s + 4)ds = \frac{1}{60}(2x^5 - 15x^4 + 40x^3 - 25x^2 - 10x + 12),$$

$$A_0^{-1}A^{-1} = -\frac{1}{12}\int_0^1 [3x^2 + 3x(1-2s) - 4s^3 + 12s^2 - 6s - 1](3s+1)ds + \tag{2.38}$$
$$+\frac{1}{2}\int_0^x (x-s)^2(3s+1)ds = \frac{1}{120}(15x^4 + 20x^3 - 75x^2 + 15x + 19).$$

Using (2.34) for $S = S(x) = 8x^2$, $G = G(x) = 3x+1$, $f(x) = 2x^2 - 6x + 4$ we find

$$A^{-1}S = \frac{1}{2}\int_0^1 \left(t - x - \frac{1}{2}\right)(8t^2)dt + \int_0^x (x-t)(8t^2)dt = \frac{2x^4 - 4x + 1}{3},$$
$$A^{-1}G = \frac{1}{2}\int_0^1 \left(t - x - \frac{1}{2}\right)(3t+1)dt + \int_0^x (x-t)(3t+1)dt = \frac{4x^2(x+1) - 10x + 1}{8},$$
$$A^{-1}f = \frac{1}{2}\int_0^1 \left(t - x - \frac{1}{2}\right)(2t^2 - 6t + 4)dt + \int_0^x (x-t)(2t^2 - 6t + 4)dt$$
$$= \frac{x^2(x^2 - 6x + 12) - 5x - 1}{6}.$$

Then by using (2.35) we arrive at

$$\Phi(A^{-1}G) = \frac{1}{8}\int_0^1 t[4t^2(t+1) - 10t + 1]dt = -\frac{31}{240},$$
$$F(G) = \int_0^1 t^2(3t+1)dt = \frac{13}{12},$$
$$\Phi(A^{-1}f) = \frac{1}{6}\int_0^1 t[t^2(t^2 - 6t + 12) - 5t - 1]dt = -\frac{1}{30},$$
$$F(f) = \int_0^1 t^2(2t^2 - 6t + 4)dt = \frac{7}{30}, \quad F(S) = \int_0^1 t^2(8t^2)dt = \frac{8}{5}.$$

Further by (2.17), (2.18) and (2.19) we find

$$\det L = \det[I_m - F(G)] = 1 - 13/12 = -1/12,$$
$$G_0 = G_0(x,y) = A^{-1}S + A^{-1}GL^{-1}F(S) = \frac{2x^4 - 4x + 1}{3} + \frac{4x^2(x+1) - 10x + 1}{8}(-12)\frac{8}{5}$$
$$= \frac{10x^4 - 144x^3 - 144x^2 + 340x - 31}{15},$$
$$\Phi(G_0) = \frac{1}{15}\int_0^1 t(10t^4 - 144t^3 - 144t^2 + 340t - 31)dt = \frac{347}{150},$$
$$\det L_0 = \det[I_m - \Phi(G_0)] = 1 - \frac{347}{150} = -\frac{197}{150}.$$

Since $\det L, \det L_0 \neq 0$, by Theorem 2.3, Problem (2.31) or (2.29) is correct. Applying (2.33) we thus have

$$A_0^{-1}G_0 = 2\int_0^1 (t-1)G_0(t)dt + \int_0^x G_0(t)dt =$$
$$= \frac{2}{15}\int_0^1 (t-1)(10t^4 - 144t^3 - 144t^2 + 340t - 31)dt +$$

$$+\frac{1}{15}\int_0^x(10t^4-144t^3-144t^2+340t-31)dt=$$
$$=-\frac{223}{75}+\frac{x(2x^4-36x^3-48x^2+170x-31)}{15}.$$

Substituting the above values into (2.9) we get the solution (2.30). □

3. Factorization of hyperbolic integro-differential equations with integral boundary conditions

Everywhere below $\overline{\Omega}=\{(x,y)\in\mathbb{R}^2:0\leqslant x,y\leqslant 1\}$.

Lemma 3.1. *Let $a(x),c(x)\in C[0,1]$, $K(y)\in C[0,1]$. Then the operator $A:C(\bar\Omega)\to C(\bar\Omega)$ corresponding to the problem:*

$$Au(t)=u'_y(x,y)+c(x)u(x,y)=f(x,y), \tag{3.1}$$

$$D(A)=\left\{u(x,y)\in C(\bar\Omega):u'_y(x,y)\in C(\bar\Omega),\ u(x,0)=a(x)\int_0^1K(y)u(x,y)dy\right\}$$

is correct if and only if

$$a(x)\int_0^1K(y)e^{-yc(x)}dy\neq 1, \tag{3.2}$$

and the unique solution of the above problem is given by the formula

$$u(x,y)=A^{-1}f(x,y)=a(x)e^{-yc(x)}\left(1-a(x)\int_0^1K(y)e^{-yc(x)}dy\right)^{-1}\times$$
$$\times\int_0^1K(y)e^{-yc(x)}\int_0^yf(x,t)e^{tc(x)}dtdy+e^{-yc(x)}\int_0^yf(x,t)e^{tc(x)}dt. \tag{3.3}$$

Proof. Assume that $u(x,y)\in\ker A$ and (3.2) hold. Then from (3.1) we get

$$u'_y(x,y)+c(x)u(x,y)=0,\quad u(x,0)=a(x)\int_0^1K(y)u(x,y)dy. \tag{3.4}$$

From the above equation by integration on y we obtain

$$u(x,y)=u(x,0)e^{-yc(x)},\quad u(x,y)=a(x)e^{-yc(x)}\int_0^1K(y)u(x,y)dy, \tag{3.5}$$
$$\int_0^1K(y)u(x,y)dy=a(x)\int_0^1K(y)e^{-yc(x)}dy\int_0^1K(y)u(x,y)dy,$$
$$\left[1-a(x)\int_0^1K(y)e^{-yc(x)}dy\right]\int_0^1K(y)u(x,y)dy=0.$$

From the last equation, since (3.2), follows that $\int_0^1K(y)u(x,y)dy=0$. Substitution of this value into (3.5) implies $u(x,y)=0$. This means that the operator A is injective.
Conversaly. Let $u(x,y)\in\ker A$ and $a(x)\int_0^1K(y)e^{-yc(x)}dy=1$. Then (3.4) holds. It is easy to verify that $u(x,y)=e^{-yc(x)}$ satisfies problem (3.4). Thus we prove that $u(x,y)=e^{-yc(x)}\in\ker A$ and so A is not injective.
We will find the solution to (3.1). Let $a(x)\int_0^1K(y)e^{-yc(x)}dy\neq 1$. Then A is injective and problem (3.1) has a unique solution. From (3.1) by integration on y we obtain

$$u(x,y)=e^{-yc(x)}a(x)\int_0^1K(y)u(x,y)dy+e^{-yc(x)}\int_0^yf(x,t)e^{tc(x)}dt, \tag{3.6}$$
$$\int_0^1K(y)u(x,y)dy=a(x)\int_0^1K(y)e^{-yc(x)}dy\int_0^1K(y)u(x,y)dy+$$
$$+\int_0^1K(y)e^{-yc(x)}\int_0^yf(x,t)e^{tc(x)}dtdy,$$
$$\left[1-a(x)\int_0^1K(y)e^{-yc(x)}dy\right]\int_0^1K(y)u(x,y)dy=$$
$$=\int_0^1K(y)e^{-yc(x)}\int_0^yf(x,t)e^{tc(x)}dtdy.$$

Then since (3.2) we obtain

$$\int_0^1K(y)u(x,y)dy=\left(1-a(x)\int_0^1K(y)e^{-yc(x)}dy\right)^{-1}\times$$
$$\times\int_0^1K(y)e^{-yc(x)}\int_0^yf(x,t)e^{tc(x)}dtdy. \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain the unique solution (3.3) to (3.1) for every $f \in C(\bar{\Omega})$. Since f in (3.3) is an arbitrary element of $C(\bar{\Omega})$, then $R(A) = C(\bar{\Omega})$. It is easy to verify that A^{-1} is bounded. Hence A is correct. □

Lemma 3.2. *Let $b(y), d(y) \in C[0,1]$, $K_0(x) \in C[0,1]$. Then the operator $A_0 : C(\bar{\Omega}) \to C(\bar{\Omega})$ corresponding to the problem:*

$$A_0 u(t) = u'_x(x,y) + d(y)u(x,y) = f(x,y), \tag{3.8}$$
$$D(A_0) = \left\{ u(x,y) \in C(\bar{\Omega}) : u'_x \in C(\bar{\Omega}),\ u(0,y) = b(y)\textstyle\int_0^1 K_0(x)u(x,y)dx \right\}$$

is correct if and only if

$$b(y)\int_0^1 K_0(x)e^{-xd(y)}dx \neq 1 \tag{3.9}$$

and the unique solution of the above problem is given by the formula

$$u(x,y) = A_0^{-1} f(x,y) = b(y)e^{-xd(y)} \left(1 - b(y)\textstyle\int_0^1 K_0(x)e^{-xd(y)}dx\right)^{-1} \times$$
$$\times \textstyle\int_0^1 K_0(x)e^{-xd(y)} \int_0^x f(s,y)e^{sd(y)}dsdx + e^{-xd(y)} \int_0^x f(s,y)e^{sd(y)}ds. \tag{3.10}$$

Proof. Assume that $u(x,y) \in \ker A_0$ and (3.9) hold. Then from (3.8) we get

$$u'_x(x,y) + d(y)u(x,y) = 0, \quad u(0,y) = b(y)\int_0^1 K_0(x)u(x,y)dx. \tag{3.11}$$

From the last equation by integration on x we obtain

$$u(x,y) = u(0,y)e^{-xd(y)}, \quad u(x,y) = e^{-xd(y)}b(y)\textstyle\int_0^1 K_0(x)u(x,y)dx, \tag{3.12}$$
$$\textstyle\int_0^1 K_0(x)u(x,y)dx = b(y)\int_0^1 K_0(x)e^{-xd(y)}dx \int_0^1 K_0(x)u(x,y)dx,$$
$$\left[1 - b(y)\textstyle\int_0^1 K_0(x)e^{-xd(y)}dx\right] \int_0^1 K_0(x)u(x,y)dx = 0.$$

If $b(y)\int_0^1 K_0(x)e^{-xd(y)}dx \neq 1$, we get $\int_0^1 K_0(x)u(x,y)dx = 0$. Substitution of this value into (3.12) implies $u(x,y) = 0$. This means that A_0 is injective.

Conversaly. Let $u(x,y) \in \ker A_0$ and $b(y)\int_0^1 K_0(x)e^{-xd(y)}dx = 1$. Then (3.11) holds. It is easy to verify that $u(x,y) = e^{-xd(y)} \neq 0$ satisfies (3.11). Thus we prove that $\ker A_0 \neq \{0\}$ and so A_0 is not injective.

We will find the solution to (3.8). Let $b(y)\int_0^1 K_0(x)e^{-xd(y)}dx \neq \pm 1$. Then A_0 is injective and Problem (3.8) has a unique solution. From (3.8) by integration on x for every $f \in C(\bar{\Omega})$ we obtain

$$u(x,y) = e^{-xd(y)}b(y)\textstyle\int_0^1 K_0(x)u(x,y)dx + e^{-xd(y)}\int_0^x f(s,y)e^{sd(y)}ds, \tag{3.13}$$
$$\textstyle\int_0^1 K_0(x)u(x,y)dx = b(y)\int_0^1 K_0(x)e^{-xd(y)}dx \int_0^1 K_0(x)u(x,y)dx +$$
$$+ \textstyle\int_0^1 K_0(x)e^{-xd(y)} \int_0^x f(s,y)e^{sd(y)}dsdx,$$
$$\left[1 - b(y)\textstyle\int_0^1 K_0(x)e^{-xd(y)}dx\right] \int_0^1 K_0(x)u(x,y)dx =$$
$$= \textstyle\int_0^1 K_0(x)e^{-xd(y)} \int_0^x f(s,y)e^{sd(y)}dsdx.$$

Then since (3.9) we obtain

$$\int_0^1 K_0(x)u(x,y)dx = \left(1 - b(y)\textstyle\int_0^1 K_0(x)e^{-xd(y)}dx\right)^{-1} \times$$
$$\times \textstyle\int_0^1 K_0(x)e^{-xd(y)} \int_0^x f(s,y)e^{sd(y)}dsdx. \tag{3.14}$$

Substituting (3.14) into (3.13), we obtain the unique solution (3.10) to (3.8) for every $f \in C(\bar{\Omega})$. Since f in (3.10) is an arbitrary element of $C(\bar{\Omega})$, then $R(A_0) = C(\bar{\Omega})$. It is easy to verify that A_0^{-1} is bounded. Hence A_0 is correct. □

Theorem 3.3. *Let $a(x), c(x), K_0(x) \in C[0,1]$, $b(y), K(y) \in C[0,1]$, $d(y) \in C^1[0,1]$, $h(x,y), u(x,y) \in C^1(\bar{\Omega})$, $u''_{xy}(x,y) \in C(\bar{\Omega})$. Then the problem*

$$u''_{xy}(x,y) + c(x)u_x(x,y) + d(y)u'_y(x,y) + h(x,y)u(x,y) = f(x,y), \tag{3.15}$$
$$u(0,y) = b(y)\textstyle\int_0^1 K_0(x)u(x,y)dx,$$
$$u'_x(x,0) + d(0)u(x,0) = a(x)\textstyle\int_0^1 K(y)[u'_x(x,y) + d(y)u(x,y)]dy$$

is correct if

$$h(x,y)=d'(y)+c(x)d(y), \tag{3.16}$$

$$a(x)\int_0^1 K(y)e^{-yc(x)}dy\neq 1,\quad b(y)\int_0^1 K_0(x)e^{-xd(y)}dx\neq 1 \tag{3.17}$$

and its unique solution is given by the formula

$$u(x,y)=b(y)e^{-xd(y)}\left(1-b(y)\int_0^1 K_0(x)e^{-xd(y)}dx\right)^{-1}\times \tag{3.18}$$

$$\times\int_0^1 K_0(x)e^{-xd(y)}\int_0^x v(s,y)e^{sd(y)}dsdx+e^{-xd(y)}\int_0^x v(s,y)e^{sd(y)}ds,$$

where

$$v(x,y)=a(x)e^{-yc(x)}\left(1-a(x)\int_0^1 K(y)e^{-yc(x)}dy\right)^{-1}\times \tag{3.19}$$

$$\times\int_0^1 K(y)e^{-yc(x)}\int_0^y f(x,t)e^{tc(x)}dtdy+e^{-yc(x)}\int_0^y f(x,t)e^{tc(x)}dt.$$

Proof. Let the operator A be defined by (3.1) and the operator A_0 by (3.8), where we suppose that $d(y)\in C^1[0,1]$. Denote by A_1 the operator corresponding to Problem (3.15), namely:

$$A_1u(x,y)=u''_{xy}(x,y)+c(x)u'_x(x,y)+d(y)u'_y(x,y)+h(x,y)u(x,y), \tag{3.20}$$

$$D(A_1)=\{u(x,y)\in C(\bar\Omega):\ u'_x(x,y),u'_y(x,y),u''_{xy}(x,y)\in C(\bar\Omega), \tag{3.21}$$

$$u(0,y)=b(y)\int_0^1 K_0(x)u(x,y)dx,$$

$$u'_x(x,0)+d(0)u(x,0)=a(x)\int_0^1 K(y)[u'_x(x,y)+d(y)u(x,y)]dy\}.$$

We will prove that $A_1=AA_0$, i.e. $D(A_1)=D(AA_0)$, $A_1u=AA_0u$ for all $u\in D(A_1)$ if $h(x,y)=d'(y)+$ $+c(x)d(y)$. Using the definition of a superposition of two operators, we find

$$D(AA_0)=\{u\in D(A_0):A_0u\in D(A)\}= \tag{3.22}$$

$$=\{u(x,y)\in C(\bar\Omega):\ u'_x\in C(\bar\Omega),\ u(0,y)=b(y)\int_0^1 K_0(x)u(x,y)dx,\ A_0u\in D(A)\}=$$

$$=\{u(x,y)\in C(\bar\Omega):\ u'_x(x,y)\in C(\bar\Omega),\ (u'_x(x,y)+d(y)u(x,y))'_y\in C(\bar\Omega),$$

$$u(0,y)=b(y)\int_0^1 K_0(x)u(x,y)dx,$$

$$u'_x(x,0)+d(0)u(x,0)=a(x)\int_0^1 K(y)[u'_x(x,y)+d(y)u(x,y)]dy\},$$

$$AA_0u(x,y)=(u'_x(x,y)+d(y)u(x,y))'_y+c(x)[u'_x(x,y)+d(y)u(x,y)]. \tag{3.23}$$

Since $d(y)\in C^1[0,1]$, from $(u'_x(x,y)+d(y)u(x,y))'_y\in C(\bar\Omega)$ follows that $u''_{xy}\in C(\bar\Omega)$ and

$$(u'_x(x,y)+d(y)u(x,y))'_y=u''_{xy}(x,y)+d'(y)u(x,y)+d(y)u'_y(x,y)\in C(\bar\Omega).$$

Then from (3.22) follows that $D(AA_0)=D(A_1)$. Furthermore if the condition (3.16) is additionally satisfied then (3.23) implies $A_1u=AA_0u$ for all $u\in D(A_1)$. Thus we proved that if (3.16) holds, then $A_1=AA_0$. Now we find the solvability condition and solution of $A_1u=f$, $u\in D(A_1)$ for the case when (3.16) holds. Denote by $v(x,y)=A_0u(x,y)=u'_x(x,y)+d(y)u(x,y)$. Then $A_1u=AA_0u=Av=f$. The last equation is correct by Lemma 3.1 if and only if (3.2) is satisfied. Then $v=A_0u=A^{-1}f$ where $A^{-1}f$ is calculated by (3.3) which is (3.19). The equation $A_0u=v$ is correct by Lemma 3.2 if and only if (3.9) is satisfied. Then $u=A_0^{-1}v$ where $A_0^{-1}v$ is calculated by (3.10) which is (3.18). Thus we proved that if (3.16), (3.17) hold true then the operator A_1 or Problem (3.15) is correct and its unique solution is (3.18) where $v(x,y)$ is given by (3.19). The theorem is proved. □

From Theorem 3.3 for $c(x)=d(y)=h(x,y)=0$ follows the next

Corollary 3.4. *Let* $a(x),K_0(x)\in C[0,1]$, $b(y),K(y)\in C[0,1]$, $u(x,y)\in C^1(\bar\Omega)$, $u''_{xy}(x,y)\in C(\bar\Omega)$. *Then the problem*

$$u''_{xy}(x,y)=f(x,y), \tag{3.24}$$

$$u(0,y)=b(y)\int_0^1 K_0(x)u(x,y)dx,$$

$$u'_x(x,0)=a(x)\int_0^1 K(y)u'_x(x,y)dy$$

is correct on $C(\bar\Omega)$ *if*

$$a(x)\int_0^1 K(y)dy\neq 1,\quad b(y)\int_0^1 K_0(x)dx\neq 1. \tag{3.25}$$

and the unique solution of Problem (3.24) is given by the formula

$$u(x,y) = \tag{3.26}$$

$$= \frac{b(y)}{1-b(y)\int_0^1 K_0(x)dx} \int_0^1 K_0(x)\left[\int_0^x \frac{a(s)}{1-a(s)\int_0^1 K(y)dy}\int_0^1 K(y)\int_0^y f(s,t)dtdyds + \right.$$

$$\left. + \int_0^x\int_0^y f(s,t)dtds\right]dx + \int_0^x \frac{a(s)}{1-a(s)\int_0^1 K(y)dy}\int_0^1 K(y)\int_0^y f(s,t)dtdyds +$$

$$+ \int_0^x\int_0^y f(s,t)dtds.$$

The following problem is solved by Theorem 2.3.

Example 3.5. Let $u(x,y), u'_x(x,y), u'_y(x,y), u''_{xy} \in C(\overline{\Omega})$. Then the problem

$$\begin{gathered} u''_{xy} \quad -(x+y)\int_0^1\int_0^1 xu'_x(x,y)dxdy - 3x^3\int_0^1\int_0^1 y^2u''_{xy}(x,y)dxdy \\ = 15x^3-2x-2y, \\ u'_x(x,0)=0, \quad u(0,y)=(y+1)\int_0^1 u(x,y)dx, \end{gathered} \tag{3.27}$$

is uniquelly solvable if $y \neq 0$ and the unique solution of (3.27) is given by the formula

$$u(x,y) = 5x^4y - y - 1. \tag{3.28}$$

Proof. Denote by B_1 the operator corresponding to Problem (3.27). First we must determine the operators A and A_0 and make sure that $D(B_1) = D(AA_0)$. Comparing (3.27) with (2.14) it is natural to take $X = C(\overline{\Omega})$,

$$\begin{gathered} \Phi(A_0u) = \int_0^1\int_0^1 xu'_x(x,y)dxdy, \quad F(AA_0u) = \int_0^1\int_0^1 y^2u''_{xy}(x,y)dxdy, \\ AA_0u(x,y) = u''_{xy}(x,y), \quad A_0u = u'_x(x,y). \end{gathered} \tag{3.29}$$

Denote $v(x,y) = u'_x(x,y)$. Then $AA_0u(x,y) = u''_{xy}(x,y) = (u'_x(x,y))'_y = Av(x,y) = v'_y(x,y)$. From boundary conditions (3.27) follws that $v(x,0)=0$. So the operators A, A_0 are defined by

$$Av(x,y) = v'_y(x,y), \quad D(A) = \{v(x,y) \in C(\bar{\Omega}) : v'_y \in C(\bar{\Omega}),\, v(x,0)=0\},$$

$$A_0u(x,y) = u'_x(x,y), \quad D(A_0) = \big\{u(x,y) \in C(\bar{\Omega}) : u'_x \in C(\bar{\Omega}),$$

$$u(0,y) = (y+1)\int_0^1 u(x,y)dx\big\}.$$

Then

$$D(AA_0) = \{u(x,y) \in D(A_0) : A_0u \in D(A)\} = \{u(x,y) \in C(\bar{\Omega}) :$$

$$u'_x, u''_{xy} \in C(\bar{\Omega}), \quad u(0,y) = (y+1)\int_0^1 u(x,y)dx,\, u'_x(x,0)=0\} = D(B_1).$$

Since $D(B_1) = D(AA_0)$, we can apply Theorem 2.3. Note that the operator A coincides with the operator A from Lemma 3.1 if $a(x) = c(x) = 0$ and the operator A_0 coincides with the operator A_0 from Lemma 3.2 if $b(y) = y+1$, $d(y) = 0$, $K_0(x) = 1$. Then by Lemma 3.1, the operator A is correct and

$$A^{-1}f(x,y) = \int_0^y f(x,t)dt, \tag{3.30}$$

by Lemma 3.2 the operator A_0 is correct if and only if $y \neq 0$ and its inverse is defined by

$$A_0^{-1}f(x,y) = -\frac{y+1}{y}\int_0^1 (1-s)f(s,y)ds + \int_0^x f(s,y)ds. \tag{3.31}$$

Notice that the operator AA_0 coincides with the operator corresponding to Problem (3.24) and, by Corollary 3.4, is correct if $y \neq 0$ and its inverse is defined by

$$A_0^{-1}A^{-1}f(x,y) = -\frac{y+1}{y}\int_0^1\int_0^x\int_0^y f(s,t)dtdsdx + \int_0^x\int_0^y f(s,t)dtds. \tag{3.32}$$

Comparing again (3.27) with (2.14) it is natural to take $S = x+y$, $G = 3x^3$, $f = 15x^3-2x-2y$. From (3.29) follows that

$$\Phi(f) = \int_0^1\int_0^1 xf(x,y)dxdy, \quad F(f) = \int_0^1\int_0^1 y^2f(x,y)dxdy. \tag{3.33}$$

Using (3.32) for $f = 15x^3-2x-2y$ and $G = 3x^3$ we find

$$A_0^{-1}A^{-1}f(x,y) = -\frac{y+1}{y}\int_0^1\int_0^x\int_0^y(15s^3-2s-2t)dtdsdx+$$
$$+\int_0^x\int_0^y(15s^3-2s-2t)dtds = \frac{45x^4y-12x^2y-12xy^2+(y+1)(6y-5)}{12},$$
$$A_0^{-1}A^{-1}G = -\frac{y+1}{y}\int_0^1\int_0^x\int_0^y 3s^3dtdsdx+$$
$$+\int_0^x\int_0^y 3s^3dtds = -\frac{3}{20}(y+1)+\frac{3}{4}x^4y+\frac{5x^4y+6x^2y+6xy^2-3(y+1)^2}{12}.$$

By means (3.30) for $G = 3x^3$, $S = x+y$ we get

$$A^{-1}G = \int_0^y G(x,t)dt = \int_0^y 3x^3dt = 3x^3y,$$
$$A^{-1}S = \int_0^y S(x,t)dt = \int_0^y (x+t)dt = xy + y^2/2,$$
$$A^{-1}f = \int_0^y f(x,t)dt = \int_0^y(15x^3-2x-2y)dt = 15x^3y-2xy-y^2.$$

Using (3.33) we get

$$F(S) = \int_0^1\int_0^1 y^2S(x,y)dxdy = \int_0^1\int_0^1 y^2(x+y)dxdy = \frac{5}{12},$$
$$F(G) = \int_0^1\int_0^1 y^2G(x,y)dxdy = \int_0^1\int_0^1 y^2 3x^3dxdy = \frac{1}{4},$$
$$F(f) = \int_0^1\int_0^1 y^2f(x,y)dxdy = \int_0^1\int_0^1 y^2(15x^3-2x-2y)dxdy = \frac{5}{12},$$
$$\Phi(A^{-1}G) = \int_0^1\int_0^1 x3x^3ydxdy = \frac{3}{10},$$
$$\Phi(A^{-1}f) = \int_0^1\int_0^1 x(15x^3y-2xy-y^2)dxdy = 1.$$

Further by (2.17)-(2.19) we find

$$\begin{aligned} L &= I_m - F(G) = 1 - 1/4 = 3/4,\\ G_0 &= G_0(x,y) = A^{-1}S + A^{-1}GL^{-1}F(S) = xy + \frac{y^2}{2} + \frac{5}{3}x^3y,\\ L_0 &= I_m - \Phi(G_0) = 1 - \int_0^1\int_0^1 xG_0(x,y)dxdy\\ &= 1 - \int_0^1\int_0^1 x\left(xy+\frac{y^2}{2}+\frac{5}{3}x^3y\right)dxdy = \frac{7}{12}. \end{aligned}$$

Since $\det L = 3/4 \neq 0$ and $\det L_0 = 7/12 \neq 0$, by Theorem 2.3, Problem 3.27 is correct. Applying (3.31) obtain

$$\begin{aligned} A_0^{-1}G_0 &= -\frac{y+1}{y}\int_0^1(1-s)G_0(s,y)ds + \int_0^x G_0(s,y)ds\\ &= \frac{5x^4y+6x^2y+6xy^2-3(y+1)^2}{12}. \end{aligned}$$

Substituting the above values into (2.9) we get (3.28). □

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Научная статья

DOI: 10.18287/2541-7525-2021-27-1-29-43

УДК 629

Дата: поступления статьи: 15.01.2021
после рецензирования: 17.02.2021
принятия статьи: 28.02.2021

Е. Провидас
Университет Фессалии, г. Ларисса, Греция
E-mail: providas@uth.gr. ORCID: https://orcid.org/0000-0002-0675-4351
Л.С. Пулькина
Самарский национальный исследовательский университет
имени академика С.П. Королева, г. Самара, Российская Федерация
E-mail: louise@samdiff.ru. ORCID: https://orcid.org/0000-0001-7947-612
И.Н. Парасидис
Университет Фессалии, г. Ларисса, Греция
E-mail: paras@teilar.gr. ORCID: https://orcid.org/0000-0002-7900-9256

ФАКТОРИЗАЦИЯ ОБЫКНОВЕННЫХ И ГИПЕРБОЛИЧЕСКИХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ИНТЕГРАЛЬНЫМИ УСЛОВИЯМИ В БАНАХОВОМ ПРОСТРАНСТВЕ

АННОТАЦИЯ

В статье исследованы условия существования единственного точного решения для одного класса абстрактных операторных уравнений вида $B_1u = \mathcal{A}u - S\Phi(A_0u) - GF(\mathcal{A}u) = f, \quad u \in D(B_1)$, где $\mathcal{A}, A_0$ — линейные абстрактные операторы; G, S — линейные векторы; Φ, F — линейные функциональные векторы. Этот класс уравнений полезен для решения краевых задач для интегро-дифференциальных уравнений в случае, когда $\mathcal{A}, A_0$ — дифференциальные операторы, а $F(\mathcal{A}u)$, $\Phi(A_0u)$ — интегральные операторы Фредгольма. Показано, что операторы типа B_1 могут быть в некоторых случаях представлены как произведения двух более простых операторов B_G, B_{G_0} специального вида, что позволяет получить условие существования единственного точного решения уравнения $B_1u = f$ из условий однозначной разрешимости уравнений $B_Gv = f$ и $B_{G_0}u = v$.

Ключевые слова: корректная (по Адамару) разрешимость; метод факторизации (декомпозиции); интегро-дифференциальные уравнения Фредгольма; начальная задача; нелокальная краевая задача с интегральными условиями.

Цитирование. Providas E., Pulkina L.S., Parasidis I.N. Factorization of ordinary and hyperbolic integro-differential equations with integral boundary conditions in a Banach space // Вестник Самарского университета. Естественнонаучная серия. 2021. Т. 27, № 1. С. 29–43. DOI: http://doi.org/10.18287/2541-7525-2021-27-1-29–43.

Информация о конфликте интересов: авторы и рецензенты заявляют об отсутствии конфликта интересов.

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Евтимиос Провидас — кандидат технических наук, доцент, Университет Фессалии, Греция, г. Ларисса, Гайополис, 41110.
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Людмила Степановна Пулькина — доктор физико-математических наук, профессор кафедры дифференциальных уравнений и теории управления, Самарский национальный исследовательский университет имени академика С.П. Королева, 443086, Российская Федерация, г. Самара, Московское шоссе, 34.
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Иван Нестерович Парасидис — кандидат технических наук, доцент, Университет Фессалии, Греция, г. Ларисса, Гайополис, 41110.